BOUNDARY VALUE AND EXPANSION PROBLEMS OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS*

BY

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Introduction.

Let $p_2(x)$, $p_3(x)$, $\cdots$, $p_n(x)$ be functions of the real variable x on the closed interval (a, b), which are continuous with their derivatives of all orders. Write

$$L(z) \equiv \frac{d^n z}{dx^n} + * + p_2(x)\frac{d^{n-2}z}{dx^{n-2}} + \cdots + p_n(x)z,$$

$$M(z) \equiv (-1)^n \frac{d^n z}{dx^n} + * + (-1)^{n-2}\frac{d^{n-2}}{dx^{n-2}}[p_2(x)z] + \cdots + p_n(x)z. \tag{1}$$

With the linear differential equation of the nth order in u

$$L(u) + \lambda u = 0 \dagger \tag{2}$$

and n linear homogeneous conditions in $u(a)$, $u'(a)$, $\cdots$, $u^{(n-1)}(a)$, $u(b)$, $u'(b)$, $\cdots$, $u^{(n-1)}(b)$,

$$W_1(u) = 0, \quad W_2(u) = 0, \quad \cdots, \quad W_n(u) = 0 \tag{3}$$

we associate the like adjoint differential equation

$$M(v) + \lambda v = 0 \tag{4}$$

and n like *adjoint conditions*

$$V_1(v) = 0, \quad V_2(v) = 0, \quad \cdots, \quad V_n(v) = 0. \tag{5}$$

For certain *characteristic values* of the complex parameter λ there will exist

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† It is not an essential generalization to write instead of (2)

$$p_0(x)\frac{d^n z}{dx^n} + p_1(x)\frac{d^{n-1}z}{dx^{n-1}} + \cdots + p_n(x)z + \lambda P(x)z = 0$$

if $p_0 \neq 0$, $P \neq 0$, and p_0/P is real. We therefore restrict (2) as stated. MAX MASON has treated a special case $n=2$ with the restriction $P \neq 0$ removed, by different methods: these Transactions, vol. 8 (1907), p. 427.

a solution* $u(x)$ of (2), (3) or $v(x)$ of (4), (5). These values are the same in both cases: let them be

$$\lambda_1, \lambda_2, \cdots$$

with corresponding solutions

$$u_1(x), u_2(x), \cdots,$$

$$v_1(x), v_2(x), \cdots.$$

These functions $u_i(x)$, $v_i(x)$ are such that

$$\int_a^b u_i(x)v_j(x)dx = 0 \qquad (i \neq j). \tag{6}$$

This property leads us to the formal expansion of a given function $f(x)$ on (a, b),

$$f(x) \sim \sum_{i=1}^{\infty} \frac{\int_a^b f(x)v_i(x)dx}{\int_a^b u_i(x)v_i(x)dx} \cdot u_i(x). \tag{7}$$

How are the characteristic values distributed in the λ-plane? What is the nature of the solutions $u_i(x)$, $v_i(x)$? In what sense does the expansion represent $f(x)$? These are the questions considered in this paper.

We begin with the derivation of the formal properties of the boundary value problem (§ 1), and of the expression for the sum of n terms of (7) by means of a contour integral (§ 2).

There follows the more intimate study which is based on certain facts concerning the asymptotic nature of the solutions of (2) and of (4) when $|\lambda|$ is large. These facts are derived as an application of my paper in a preceding number of these Transactions (§ 3). The distribution of the numbers λ_i and the nature of the expansion is then obtained (§ 4 and § 5). Finally the contour integral is evaluated and the representation theorem proved (§ 6). The expansion is found to behave like a Fourier series except in the vicinity of $x = a$ and $x = b$.

LIOUVILLE was the first to introduce the notion of adjoint conditions in a special case † and to consider the related expansion. The results of the present paper are known for the *real self-adjoint case* $n = 2$.‡ WESTFALL has proved a representation theorem for the real self-adjoint case $n = 2m$, providing that $f(x)$ and its first n derivatives are continuous. §

* By a solution we mean always a solution not identically zero.

† Liouville's Journal, ser. 1, vol. 3 (1838), p. 561. Professor E. H. MOORE suggested to me the possibility of generalization.

‡ A. KNESER, Mathematische Annalen, vol. 58 (1901), p. 81; DIXON, Proceedings of the London Mat' ematical Society, ser. 2, vol. 3 (1905), p. 83.

§ W. D. A. WESTFALL, *Zur Theorie der Integralgleichungen*, A. MYLLER, *Gewöhnliche Differentialgleichungen höherer Ordnung in ihrer Beziehung zu den Integralgleichungen*. Göttingen dissertations (1905 and 1906). Reference should also be made to HILBERT's antecedent papers on integral equations in the Göttinger Nachrichten for 1904.

§ 1. *The formal nature of the boundary value problem.*

We must first make precise the adjoint conditions referred to.

DEFINITION. Let $W_1(u), \cdots, W_n(u)$ be n given linear forms in $u(a)$, $u'(a), \cdots, u^{(n-1)}(a), u(b), u'(b), \cdots, u^{(n-1)}(b)$ and $W_{n+1}(u), \cdots, W_{2n}(u)$ be any n further linear forms so chosen that $W_1, \cdots, W_{2n}$ are linearly independent. Then in the identity

$$\int_a^b zL(y)\,dx \equiv P(y,z)\Big|_{x=a}^{x=b} + \int_a^b yM(z)\,dx,^* \tag{8}$$

where $P(y,z)|_{x=a}^{x=b}$ is a bilinear form in $y(a), y'(a), \cdots, y^{(n-1)}(a), y(b), y'(b), \cdots, y^{(n-1)}(b)$ and $z(a), z'(a), \cdots, z^{(n-1)}(a), z(b), z'(b), \cdots, z^{(n-1)}(b)$, we can write

$$P(y,z)\Big|_{x=a}^{x=b} \equiv \sum_{i=1}^{2n} W_i(y)\,V_{2n-i}(z), \tag{9}$$

in which $V_i(z)$ are linear in $z(a), z'(a), \cdots, z^{(n-1)}(a), z(b), z'(b), \cdots, z^{(n-1)}(b)$ and linearly independent. Then $V_i(v)=0$ $(i=1, 2, \cdots, n)$ are the adjoint conditions to the given conditions $W_i(u)=0$ $(i=1, 2, \cdots, n)$. Any set of conditions $\overline{W}_i(u)=0$ $(i=1, 2, \cdots, n)$ equivalent to $W_i(u)=0$ $(i=1, 2, \cdots, n)$ by linear combination, with any choice of $\overline{W}_{n+1}(u), \cdots, \overline{W}_{2n}(u)$ leads to a set of adjoint conditions $\overline{V}_i(v)=0$ $(i=1, 2, \cdots, n)$ equivalent to $V_i(v)=0$ $(i=1, 2, \cdots, n)$ by linear combination.

Conversely, given $V_i(v)=0\,(i=1, 2, \cdots, n)$ we choose $V_{n+1}, \cdots, V_{2n}$ as above and find $W_i(u)=0\,(i=1, 2, \cdots, n)$ to be the adjoint conditions. Hence if the problem (4), (5) is adjoint to (2), (3), so also is (2), (3) adjoint to (4), (5).

The properties which this section proves are stated in I–III.

I. *If for $\lambda=\lambda^*$ a solution $u^*(x)$ of* (2), (3) *exists, a solution $v^*(x)$ of* (4), (5) *will also exist for $\lambda=\lambda^*$; if $u^*(x)$ is unique (except for a constant factor), $v^*(x)$ is also unique (except for a constant factor).*

Let u^* be the given solution of (2), (3) for $\lambda=\lambda^*$. Then we have

$$W_1(u^*) = W_2(u^*) = \cdots = W_n(u^*) = 0,$$

and for some j

$$W_{n+j}(u^*) \neq 0,$$

as otherwise we should infer

$$u^*(a) = u^{*\prime}(a) = \cdots = u^{*(n-1)}(a) = u^*(b) = u^{*\prime}(b) = \cdots = u^{*(n-1)}(b) = 0.$$

In the n-fold linear spread of solutions $v(x)$ of (4) at $\lambda=\lambda^*$ there will be at least one, say $v^*(x)$, which satisfies the $n-1$ linear homogeneous conditions

$$V_{n-i}(v^*) = 0 \qquad (i \neq j).$$

* Cf. SCHLESINGER: *Handbuch der Theorie der linearen Differentialgleichungen*, vol. 1, p. 54, formula (1).

Substitute in (8) $y = u^*$, $z = v^*$ and we obtain

$$-\lambda^* \int_a^b v^* u^* dx = W_{n+j}(u^*)\, V_{n-j}(v^*) - \lambda^* \int_a^b u^* v^* dx.$$

Therefore $V_{n-j}(v^*) = 0$ also, and v^* will satisfy (4), (5).

If u^* is unique, v^* is also unique. To prove this assume if possible that there were two linearly independent solutions v^* and v^{**} satisfying (4), (5) when $\lambda = \lambda^*$, while u^* is unique. Then we could choose i and j so that

$$\begin{vmatrix} V_{n+i}(v^*) & V_{n+j}(v^*) \\ V_{n+i}(v^{**}) & V_{n+j}(v^{**}) \end{vmatrix} \neq 0;$$

otherwise we should have constants c, d, not both zero, such that

$$V_{n+l}(cv^* + dv^{**}) = 0 \qquad (l = 1, 2, \cdots, n),$$

and $v = cv^* + dv^{**}$ would fulfill the $2n$ conditions

$$V_1(v) = V_2(v) = \cdots = V_{2n}(v) = 0,$$

which is not possible since the $V_1, \cdots, V_{2n}$ are linearly independent. Choose now u^{**} linearly independent of u^* to fulfill the $n - 2$ conditions

$$W_l(u^{**}) = 0, \qquad (l \neq n - i \text{ or } n - j)$$

where $l = 1, 2, \cdots, n$. Writing in (8) $y = u^{**}$, $z = v^*$ and $y = u^{**}$, $z = v^{**}$, we obtain, on simplifying,

$$W_{n-i}(u^{**})\, V_{n+i}(v^*) + W_{n-j}(u^{**})\, V_{n+j}(v^*) = 0,$$
$$W_{n-i}(u^{**})\, V_{n+i}(v^{**}) + W_{n-j}(u^{**})\, V_{n+j}(v^{**}) = 0.$$

From this we deduce that also

$$W_{n-i}(u^{**}) = W_{n-j}(u^{**}) = 0.$$

Therefore u^{**} would satisfy (2), (3). This is impossible. Hence if u^* is unique, v^* is also unique, which we were to show.

DEFINITION. If $\lambda = \lambda^*$ is a characteristic value of λ for which one and but one solution of (2), (3) and (4), (5) exists, λ^* is said to be a *simple* characteristic value.

II. *If $y_1, y_2, \cdots, y_n$ are n linearly independent solutions of* (2) *at $\lambda = \lambda^*$, the condition that λ^* is a characteristic value is that the determinant*

$$\Delta = \begin{vmatrix} W_1(y_1) & W_1(y_2) & \cdots & W_1(y_n) \\ W_2(y_1) & W_2(y_2) & \cdots & W_2(y_n) \\ \cdots & \cdots & \cdots & \cdots \\ W_n(y_1) & W_n(y_2) & \cdots & W_n(y_n) \end{vmatrix}$$

vanishes; the condition that λ^* is a simple characteristic value is that not all the first minors vanish.

The general solution of (2) is

$$u = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n .$$

Under the condition $\Delta = 0$ at least one determination of $c_1, c_2, \cdots, c_n$ is possible such that

$$W_1(u) = 0, \ W_2(u) = 0, \ \cdots, \ W_n(u) = 0 .$$

This determination is unique when not all the first minors of Δ vanish.

III. *If $u_i(x)$ and $v_j(x)$ belong to distinct characteristic numbers λ_i and λ_j, then is*

$$\int_a^b u_i v_j dx = 0 .$$

For write $y = u_i$, $z = v_j$ in (8). Because of (9) we deduce

$$-\lambda_i \int_a^b v_j u_i dx = -\lambda_j \int_a^b u_i v_j dx ,$$

whence the desired relation follows.

In important particular cases the problem (4), (5) may be precisely (2), (3) with v in place of u. We then say that the problem (2), (3) is *self-adjoint.* Or (4), (5) may be precisely (2), (3) with v in place of u, except that $M(v)$ is the *negative* of $L(v)$. We then say that the problem (2), (3) is *anti-self-adjoint.*

§ 2. *Expansion as contour integral.*

It is easy to prove that when λ is not a characteristic number there exists a unique $G(x, s; \lambda)$* such that the solution ϕ of

$$L(\phi) + \lambda\phi = \omega, \qquad W_1(\phi) = W_2(\phi) = \cdots = W_n(\phi) = 0 ,$$

is given by

$$\phi = \int_a^b G(x, s; \lambda)\, \omega(s) ds ;$$

and likewise that there exists a unique $H(x, s; \lambda)$ such that the solution ϕ of

$$M(\phi) + \lambda\phi = \omega, \qquad V_1(\phi) = V_2(\phi) = \cdots = V_n(\phi) = 0 ,$$

is given by

$$\phi = \int_a^b H(x, s; \lambda)\omega(s) ds .$$

* Professor BÔCHER has defined the Green's function G for an ordinary linear differential equation of order n under a special form of conditions, and has stated the principal properties in the Bulletin of the American Mathematical Society, vol. 7 (1901), p. 297.

The explicit formula for G is

$$(10)\quad G(x,s;\lambda)=\frac{N(x,s;\lambda)}{\Delta(\lambda)}=\frac{\begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) & \bar{G}(x,s;\lambda)\\ W_1(y_1) & W_1(y_2) & \cdots & W_1(y_n) & W_1(\bar{G})\\ W_2(y_1) & W_2(y_2) & \cdots & W_2(y_n) & W_2(\bar{G})\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ W_n(y_1) & W_n(y_2) & \cdots & W_n(y_n) & W_n(\bar{G}) \end{vmatrix}}{(-1)^n\begin{vmatrix} W_1(y_1) & W_1(y_2) & \cdots & W_1(y_n)\\ W_2(y_1) & W_2(y_2) & \cdots & W_2(y_n)\\ \cdots & \cdots & \cdots & \cdots\\ W_n(y_1) & W_n(y_2) & \cdots & W_n(y_n) \end{vmatrix}},$$

in which

$$(11)\quad \bar{G}(x,s;\lambda)=\pm\tfrac{1}{2}\cdot\frac{\begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x)\\ y_1^{(n-2)}(s) & y_2^{(n-2)}(s) & \cdots & y_n^{(n-2)}(s)\\ \cdots & \cdots & \cdots & \cdots\\ y_1(s) & y_2(s) & \cdots & y_n(s) \end{vmatrix}}{\begin{vmatrix} y_1^{(n-1)}(s) & y_2^{(n-1)}(s) & \cdots & y_n^{(n-1)}(s)\\ y_1^{(n-2)}(s) & y_2^{(n-2)}(s) & \cdots & y_n^{(n-2)}(s)\\ \cdots & \cdots & \cdots & \cdots\\ y_1(s) & y_2(s) & \cdots & y_n(s) \end{vmatrix}}$$

$$(+\text{ if } x>s,\ -\text{ if } x<s).$$

Here $y_1, y_2, \cdots, y_n$ are any n linearly independent solutions of (2). There is a like formula for $H(x,s;\lambda)$. Furthermore

$$G(x,s;\lambda)=H(s,x;\lambda).*$$

The function G is analytic in λ (since $y_1, y_2, \cdots, y_n$ may be taken analytic in λ) *except for a possible pole when* $\Delta(\lambda)=0$, i. e. *by* II, § 1, *when* λ *is a characteristic value.*

If $\lambda=\lambda_i$ *is a simple characteristic number for which* G *has a pole of the first order, the residue is*

$$\frac{u_i(x)v_i(s)}{\int_a^b u_i(x)v_i(x)\,dx},$$

where

$$\int_a^b u_i(x)v_i(x)\,dx\neq 0.$$

Write

$$(12)\quad G(x,s;\lambda)=\frac{R(x,s)}{\lambda-\lambda_i}+\sigma(x,s;\lambda),$$

* For formulas like these and their proofs see WESTFALL's dissertation, §§ 6, 7.

where $\sigma(x, s; \lambda)$ is analytic at $\lambda = \lambda_i$. In view of (10) we have

$$R(x, s) = \frac{N(x, s; \lambda_i)}{\Delta'(\lambda_i)},$$

where N is continuous with its first n derivatives since the coefficient of $\bar{G}$ is zero. Further N satisfies (2), (3) for all λ so that $R(x, s)$ must be a solution of (2), (3) in x. Because of the relation between G and H, we infer also that $R(x, s)$ is a solution of (4), (5) in s. By the definition of a simple characteristic value it follows that the residue is

$$R(x, s) = c_i u_i(x) v_i(s).*$$

It remains to determine c_i.

By (12) we have

$$\lim_{\lambda=\lambda_i} [(\lambda - \lambda_i) G(x, s; \lambda) - c_i u_i(x) v_i(s)] = 0,$$

whence

$$\lim_{\lambda=\lambda_i} \left[(\lambda - \lambda_i) \int_a^b G(x, s; \lambda) u_i(s) ds - c_i u_i(x) \int_a^b v_i(s) u_i(s) ds \right] = 0.$$

But since

$$L(u_i) + \lambda u_i = (\lambda - \lambda_i) u_i, \qquad W_1(u_i) = W_2(u_i) = \cdots = W_n(u_i) = 0,$$

we have by the fundamental property of G

$$(\lambda - \lambda_i) \int_a^b G(x, s; \lambda) u_i(s) ds = u_i(x).$$

Substituting this value above we find

$$\lim_{\lambda=\lambda_i} \left[u_i(x) - c_i u_i(x) \int_a^b v_i(s) u_i(s) ds \right] = 0.$$

Therefore

$$c_i \int_a^b v_i(s) u_i(s) ds = 1,$$

out of which c_i is determined; this proves that the residue has the stated form.

If Γ be a contour in the λ-plane which encloses $\lambda_1, \lambda_2, \cdots, \lambda_n$, we conclude by the above that

$$(13) \qquad \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \int_a^b G(x, s; \lambda) f(s) ds d\lambda = \sum_{i=1}^{n} \frac{\int_a^b f(x) v_i(x) dx}{\int_a^b u_i(x) v_i(x) dx} \cdot u_i(x).$$

* In this proof certain points of logic are obviously slurred over.

This is the sum of n terms of the formal expansion, *providing that* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *are simple characteristic numbers at which* G *has a pole of the first order.* At values λ_i for which these conditions are not satisfied, the corresponding term of the formal expansion (7) is to be replaced by

$$\int_a^b R_i(x, s) f(s) ds$$

where $R_i(x, s)$ is the residue. Thus in all cases we have an expansion each term of which is uniquely determined. We omit the development of $R_i(x, s)$ in these more complicated cases.

§ 3. *The solutions of* $L(u) + \lambda u = 0$ *and* $M(v) + \lambda v = 0$ *when* $|\lambda|$ *is large.*

As a preliminary to the deeper study of the problems which have been presented, I apply the results of my paper printed on pages 219–231 of the present volume to the differential equations $L(u) + \lambda u = 0$ and $M(v) + \lambda v = 0$.

If we write $\lambda = \rho^n$, the first of these differential equations can be written in the form there treated [see (1), loc. cit.],

$$\frac{d^n u}{dx^n} + \rho a_{n-1}(x, \rho) \frac{d^{n-1} u}{dx^{n-1}} + \cdots + \rho^n a_0(x, \rho) u = 0,$$

$$a_{n-1}(x, \rho) = 0, \; a_{n-2}(x, \rho) = \frac{p_2(x)}{\rho^2}, \; \cdots, \; a_0(x, \rho) = 1 + \frac{p_n(x)}{\rho^n}.$$

The coefficients obviously satisfy the restriction imposed, and the equation for $w_i(x)$ [see (4), loc. cit.], is

$$w_i^n + 1 = 0. \tag{14}$$

Thus $w_1, w_2, \cdots, w_n$ are constants. We also find (proof of Lemma 1, loc. cit.)

$$u_{i0}(x) = 1 \qquad (i = 1, 2, \cdots, n). \tag{15}$$

In order to state in explicit form the theorem for this case, it remains to consider the regions S (definition, p. 220). Inasmuch as $w_1, w_2, \cdots, w_n$ are constants we can for each ρ choose the indices 1 to n so that

$$R(\rho w_1) \leqq R(\rho w_2) \leqq \cdots \leqq R(\rho w_n)$$

for every x on (a, b). Hence every ρ is on some region S, $\theta \leqq \arg \rho \leqq \psi$. At the bounding rays the ordering changes so that for some i and j

$$R(\rho w_i) = R(\rho w_j) \qquad (i \neq j).$$

An easy computation shows that the regions S are the regions

$$\frac{l\pi}{n} \leqq \arg \rho \leqq \frac{(l+1)\pi}{n} \qquad (l = 0, 1, \ldots, 2n-1). \tag{16}$$

The differential equation $M(v) + \lambda v = 0$ also is of the same type when we put $\lambda = \rho^n$. The roots $w_i(x)$ are

$$-w_1, \ -w_2, \ \cdots, \ -w_n \tag{17}$$

and in the formal solutions we have

$$u_{i0}(x) = 1 \qquad (i = 1, 2, \cdots, n). \tag{18}$$

The regions S are the same as for $L(u) + \lambda u = 0$.

Let E be a generic notation for functions of ρ (and other variables), bounded when $|\rho|$ is large. The application of the theorem referred to gives:

On any region of the ρ-plane

$$S: \qquad \frac{l\pi}{n} \leqq \arg \rho \leqq \frac{(l+1)\pi}{n} \tag{19}$$

there exist n independent solutions,

$$\begin{aligned} &y_1, \ y_2, \ \cdots, \ y_n \text{ of } L(u) + \rho^n u = 0, \\ &z_1, \ z_2, \ \cdots, \ z_n \text{ of } M(v) + \rho^n v = 0, \end{aligned} \tag{20}$$

analytic in ρ and such that on this region

$$\left\{ \begin{aligned} y_i &= u_i(x, \rho) + e^{\rho w_i (x-a)} \frac{E_0}{\rho^m}, \\ \frac{dy_i}{dx} &= \frac{d}{dx} u_i(x, \rho) + e^{\rho w_i (x-a)} \frac{E_1}{\rho^{m-1}}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{d^{n-1} y_i}{dx^{n-1}} &= \frac{d^{n-1}}{dx^{n-1}} u_i(x, \rho) + e^{\rho w_i (x-a)} \frac{E_{n-1}}{\rho^{m-n+1}}, \end{aligned} \right. \tag{21}$$

$$\left\{ \begin{aligned} z_i &= v_i(x, \rho) + e^{-\rho w_i (x-a)} \frac{E_0}{\rho^m}, \\ \frac{dz_i}{dx} &= \frac{d}{dx} v_i(x, \rho) + e^{-\rho w_i (x-a)} \frac{E_1}{\rho^{m-1}}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{d^{n-1} z_i}{dx^{n-1}} &= \frac{d^{n-1}}{dx^{n-1}} v_i(x, \rho) + e^{-\rho w_i (x-a)} \frac{E_{n-1}}{\rho^{m-n+1}}, \end{aligned} \right. \tag{22}$$

where

$$
\begin{aligned}
u_i(x,\rho) &= e^{\rho w_i(x-a)}\left[1+\frac{u_{i1}(x)}{\rho}+\cdots+\frac{u_{im}(x)}{\rho^m}\right],\\
v_i(x,\rho) &= e^{-\rho w_i(x-b)}\left[1+\frac{v_{i1}(x)}{\rho}+\cdots+\frac{v_{im}(x)}{\rho^m}\right],
\end{aligned}
\tag{23}
$$

$w_1, w_2, \cdots, w_n$ being the n roots of $w^n+1=0$, m any positive integer.

§ 4. *Distribution of the characteristic values.*

The condition that λ^* is a characteristic number is that the determinant Δ vanishes (II, § 1). The i-th element in the j-th column of Δ is $W_i(y_j)$ where $y_1, y_2, \cdots, y_n$ are any n linearly independent solutions of $L(u)+\lambda^* u=0$. In order to treat the equation $\Delta=0$ we take y_i to be the y_i of (20) § 3 where $\lambda=\rho^n$. In addition we assume the conditions W_i to be normalized as follows. Reduce the number of conditions $W_i(u)=0$ of order $n-1$ [i. e., containing either $u^{(n-1)}(a)$ or $u^{(n-1)}(b)$] to a minimum, at most 2, by linear combination. Then, in those that remain reduce the number of order $n-2$ to a minimum, at most 2, by linear combination. Continue in this way as long as conditions remain. The normalized conditions will have the form

$$
\begin{aligned}
W_i(u) &\equiv W_{ia}(u)+W_{ib}(u)=0,\\
W_{ia}(u) &\equiv \alpha_i u^{(k_i)}(a)+\sum_{j=0}^{k_i-1}\alpha_{ij}u^{(j)}(a),\\
&\qquad\qquad (k_1\geqq k_2\geqq\cdots\geqq k_n),\\
W_{ib}(u) &\equiv \beta_i u^{(k_i)}(b)+\sum_{j=0}^{k_i-1}\beta_{ij}u^{(j)}(b),
\end{aligned}
\tag{24}
$$

in which no three successive k's are equal.

DEFINITION. Let the w_i be taken in every order such that for some $\rho\neq 0$

$$R(\rho w_1)<R(\rho w_2)<\cdots<R(\rho w_n).$$

If $n=2\mu-1$ and always neither $\theta_0=0$ nor $\theta_1=0$, where θ_0 and θ_1 are defined by the identity

$$
\theta_0+\theta_1 s\equiv\begin{vmatrix}
\alpha_1 w_1^{k_1} & \cdots & \alpha_1 w_{\mu-1}^{k_1} & (\alpha_1+s\beta_1)w_\mu^{k_1} & \beta_1 w_{\mu+1}^{k_1} & \cdots & \beta_1 w_n^{k_1}\\
\alpha_2 w_1^{k_2} & \cdots & \alpha_2 w_{\mu-1}^{k_2} & (\alpha_2+s\beta_2)w_\mu^{k_2} & \beta_2 w_{\mu+1}^{k_2} & \cdots & \beta_2 w_n^{k_2}\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\alpha_n w_1^{k_n} & \cdots & \alpha_n w_{\mu-1}^{k_n} & (\alpha_n+s\beta_n)w_\mu^{k_n} & \beta_n w_{\mu+1}^{k_n} & \cdots & \beta_n w_n^{k_n}
\end{vmatrix},
\tag{25}
$$

or if $n=2\mu$ and always neither $\theta_1=0$ nor $\theta_2=0$, where θ_0, θ_1, and θ_2 are defined by the identity

$$(26)\quad \theta_0+\theta_1 s+\frac{\theta_2}{s}\equiv$$

$$\begin{vmatrix} \alpha_1 w_1^{k_1} & \cdots & \alpha_1 w_{\mu-1}^{k_1} & (\alpha_1+s\beta_1)w_\mu^{k_1} & \left(\alpha_1+\frac{1}{s}\beta_1\right)w_{\mu+1}^{k_1} & \beta_1 w_{\mu+2}^{k_1} & \cdots & \beta_1 w_n^{k_1} \\ \alpha_2 w_1^{k_2} & \cdots & \alpha_2 w_{\mu-1}^{k_2} & (\alpha_2+s\beta_2)w_\mu^{k_2} & \left(\alpha_2+\frac{1}{s}\beta_2\right)w_{\mu+1}^{k_2} & \beta_2 w_{\mu+2}^{k_2} & \cdots & \beta_2 w_n^{k_2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_n w_1^{k_n} & \cdots & \alpha_n w_{\mu-1}^{k_n} & (\alpha_n+s\beta_n)w_\mu^{k_n} & \left(\alpha_n+\frac{1}{s}\beta_n\right)w_{\mu+1}^{k_n} & \beta_n w_{\mu+2}^{k_n} & \cdots & \beta_n w_n^{k_n} \end{vmatrix},$$

the conditions $W_i(u)=0$ $(i=1, 2, \cdots, n)$ are said to be *regular.**

We now proceed to prove the

THEOREM. *If the conditions $W_i(u)=0$ $(i=1, 2, \cdots, n)$ are regular, the characteristic values of λ for (2), (3) and (4), (5) are in general simple† and form a pair of infinite sequences λ_{Il}, λ_{IIl} $(l=1, 2, \cdots)$ such that*

$$(27)\quad \begin{aligned} \lambda_{Il} &= -\left(\frac{2l\pi\sqrt{-1}}{b-a}\right)^n\left(1+\sum_{j=1}^{m-1}\frac{g_{Ij}}{l^j}+\frac{E_{Il}}{l^m}\right), \\ \lambda_{IIl} &= -\left(\frac{-2l\pi\sqrt{-1}}{b-a}\right)^n\left(1+\sum_{j=1}^{m-1}\frac{g_{IIj}}{l^j}+\frac{E_{IIl}}{l^m}\right), \end{aligned}$$

where g_{Ij}, g_{IIj} are constants and $|E_{Il}|<M$, $|E_{IIl}|<M$.

We consider the equation $\Delta=0$ in the ρ-plane, fixing attention on some region (19) for $l=l_0$. The transformation $\lambda=\rho^n$ makes two adjacent regions S correspond to the entire λ-plane. It is convenient to consider separately the cases $n=2\mu-1$ and $n=2\mu$. The starting point is the set of relations [deduced from (21)]

$$(28)\quad \begin{aligned} W_{ia}(y_j) &= (\rho w_j)^{k_i}\left[\alpha_i+\sum_{l=1}^{m-1}\frac{\alpha_{ijl}}{\rho^l}+\frac{E}{\rho^m}\right], \\ W_{ib}(y_j) &= e^{\rho w_j(b-a)}(\rho w_j)^{k_i}\left[\beta_i+\sum_{l=1}^{m-1}\frac{\beta_{ijl}}{\rho^l}+\frac{E}{\rho^m}\right]. \end{aligned}$$

* This apparently gives one condition for each region S, but these reduce to two for $n=2\mu-1$ and to one when $n=2\mu$. It is worthy of note that the STURM-LIOUVILLE boundary conditions $(n=2)$ of the form

$$hu(a)+ku'(a)=0, \qquad lu(b)+mu'(b)=0,$$

and also the periodic boundary conditions $(n=2)$

$$u(a)=u(b), \qquad u'(a)=u'(b),$$

are regular. An example of non-regular conditions $(n=2)$ is

$$u(a)=0, \qquad u'(a)=ku(b).$$

† See § 1.

CASE 1. $n = 2\mu - 1$.

By definition of the regions S we can choose indices 1 to n so that

$$R(\rho w_1) \leqq R(\rho w_2) \leqq \cdots \leqq R(\rho w_n)$$

for the region (19) under consideration. Making this choice we find readily that on S

$$R(\rho w_i) < 0 \qquad (i = 1, 2, \cdots, \mu - 1),$$

(29)

$$R(\rho w_i) > 0 \qquad (i = \mu + 1, \mu + 2, \cdots, n),$$

while

$$R(\rho w_\mu) = 0, \qquad \arg \rho = \frac{(l_0 + \frac{1}{2})}{n}\pi. \tag{30}$$

From (28) and (29) we see that in the formula

$$W_i(y_j) = W_{ia}(y_j) + W_{ib}(y_j) \qquad (j = 1, 2, \cdots, \mu - 1),$$

the term $W_{ib}(y_j)$ can be absorbed into the term $W_{ia}(y_j)$ since $e^{\rho w_j(b-a)}$ is small of an infinite order in $1/\rho$ on S; that is, we can write

$$W_i(y_j) = (\rho w_j)^{k_i}\left[\alpha_i + \sum_{l=1}^{m-1}\frac{\alpha_{ijl}}{\rho^l} + \frac{E}{\rho^m}\right] \quad (j = 1, 2, \cdots, \mu - 1). \tag{31}$$

By like reasoning we find

$$W_i(y_j) = e^{\rho w_j(b-a)}(\rho w_j)^{k_i}\left[\beta_i + \sum_{l=1}^{m-1}\frac{\beta_{ijl}}{\rho^l} + \frac{E}{\rho^m}\right] \quad (j = \mu + 1, \mu + 2, \cdots, n). \tag{32}$$

Also from (28)

$$W_i(y_\mu) = (\rho w_\mu)^{k_i}\left[\alpha_i + \sum_{l=1}^{m-1}\frac{\alpha_{i\mu l}}{\rho^l} + \frac{E}{\rho^m}\right] + e^{\rho w_\mu(b-a)}(\rho w_\mu)^{k_i}\left[\beta_i + \sum_{l=1}^{m-1}\frac{\beta_{i\mu l}}{\rho^l} + \frac{E}{\rho^m}\right]. \tag{33}$$

If we substitute in Δ these values of $W_i(y_j)$ as given in (31), (32), (33), and remove the factors ρ^{k_i} from the i-th row ($i = 1, 2, \cdots, n$), and the factors $e^{\rho w_j(b-a)}$ from the j-th column ($j = \mu + 1, \mu + 2, \cdots, n$), i. e., altogether the non-zero factor

$$\prod_{i=1}^{n}\rho^{k_i} \cdot \prod_{j=\mu+1}^{n} e^{\rho w_j(b-a)},$$

the condition takes the form

$$\bar{\Delta} = \left(\theta_0 + \sum_{j=1}^{m-1}\frac{c_j}{\rho^j} + \frac{E_0}{\rho^m}\right) + e^{\rho w_\mu(b-a)}\left(\theta_1 + \sum_{j=1}^{m-1}\frac{d_j}{\rho^j} + \frac{E_1}{\rho^m}\right) = 0, \tag{34}$$

$$\Delta = \prod_{i=1}^{n}\rho^{k_i} \cdot \prod_{j=\mu+1}^{n} e^{\rho w_j(b-a)} \cdot \bar{\Delta}.$$

The quantities θ_0 and θ_1 are the θ_0 and θ_1 of (25). In fact (25) was formed to give these terms.

Since Δ was analytic in ρ, $\bar{\Delta}$ will be also analytic in ρ.

The expressions of the first minors of the μ-th column of $\bar{\Delta}$,

$$a + \frac{b}{\rho} + \cdots + \frac{E}{\rho^m},$$

cannot have all their first terms $a = 0$ since then $\theta_0 = \theta_1 = 0$, contradicting the hypothesis. Hence for large values of $|\rho|$ not all these first minors vanish. If we return to Δ, we see that accordingly not all its first minors vanish. The characteristic numbers when $|\lambda|$ is large must therefore be simple. This was part of the theorem.

From (34) we obtain the equivalent condition

$$e^{\rho w_\mu (b-a)} = \delta(\rho), \qquad \delta(\rho) = -\frac{\theta_0}{\theta_1} + \sum_{j=1}^{m-1} \frac{\delta_j}{\rho^j} + \frac{E}{\rho^m}$$

by solving for $e^{\rho w_\mu (b-a)}$. From this we infer

$$\rho w_\mu (b-a) = \log \delta(\rho), \tag{35}$$

$$\log \delta(\rho) = \log\left(-\frac{\theta_0}{\theta_1}\right) + \sum_{j=1}^{m-1} \frac{e_j}{\rho^m} + \frac{E}{\rho^m} \pm 2l\pi\sqrt{-1},$$

the final form of the condition, in which of course E is analytic in ρ on S.

In view of this condition, it is obvious that the values of ρ on S which satisfy $\Delta = 0$ have when $|\rho|$ is large the form

$$\rho_l = \pm\frac{2l\pi\sqrt{-1}}{w_\mu(b-a)} + \frac{\log\left(-\frac{\theta_0}{\theta_1}\right)}{\omega_\mu(b-a)} + \sum_{j=1}^{m-1} \frac{h_j}{l^j} + \frac{E}{l^m}, \tag{36}$$

where l is a large positive integer. Since by (30)

$$\arg \omega_\mu = -\frac{(l_0 + \frac{1}{2})}{n}\pi \pm \frac{\pi}{2},$$

we see from (36) that the $\pm$ sign is to be so chosen that the ρ_l approach asymptotically (in an angular sense) the bisecting ray of S. Let now ρ describe a circle of fixed small radius r about

$$\pm\frac{2l\pi\sqrt{-1}}{w_\mu(b-a)} + \frac{\log\left(-\frac{\theta_0}{\theta_1}\right)}{w_\mu(b-a)}.$$

This will be wholly within S when $|\rho|$ is large; and

$$\arg\left[\rho\omega_\mu(b-a) - \log\delta(\rho)\right]$$

increases by 2π if the inequality

$$|r(b-a)|>\left|\sum_{j=1}^{m-1}\frac{e_j}{\rho^j}+\frac{E}{\rho^m}\right|,$$

obtains as is true for $|\rho|$ large. *This proves the unique existence of ρ_l when $|\rho|$ is large*, and the distribution of values ρ is determined.

It remains to return to the λ-plane; S becomes S'

$$S': \qquad l_0\pi \leqq \arg\lambda \leqq (l_0+1)\pi.$$

The values of λ approach the positive or negative axis of imaginaries according as l_0 is even or odd, and by (36) have the form stated in the theorem where for some θ_0, θ_1

$$g_{\text{II}} \text{ or } g_{\text{III}} = \pm\frac{n\log\left(-\dfrac{\theta_0}{\theta_1}\right)}{2\pi\sqrt{-1}}.$$

This completes the proof when $n=2\mu-1$.

CASE II. $n=2\mu$.

The proof is analogous to that for Case 1. If $\rho w_1, \rho w_2, \cdots, \rho w_n$ have increasing real parts on S we find

$$R(\rho w_i)<0 \qquad (i=1,2,\cdots,\mu-1),$$

$$R(\rho w_i)>0 \qquad (i=\mu+2,\mu+3,\cdots,n),$$

while

$$R(\rho w_\mu)=R(\rho w_{\mu+1})=0, \qquad \arg\rho=\text{either}\ \frac{l_0\pi}{n}\ \text{or}\ \frac{(l_0+1)\pi}{n},$$

and $w_{\mu+1}=-w_\mu$. The region S is of course defined by (19) as before. We now find $W_i(y_j)$ to be of the form (31) for $j=1,2,\cdots,\mu-1$ and of the form (32) for $j=\mu+2,\mu+3,\cdots,n$; also for $j=\mu,\mu+1$ we find

$$W_i(y_\mu)=(\rho w_\mu)^{k_i}\left[\alpha_i+\sum_{l=1}^{m-1}\frac{\alpha_{i\mu l}}{\rho^l}+\frac{E}{\rho^m}\right]+e^{\rho w_\mu(b-a)}(\rho w_\mu)^{k_i}\left[\beta_i+\sum_{l=1}^{m-1}\frac{\beta_{i\mu l}}{\rho^l}+\frac{E}{\rho^m}\right],$$

(37)

$$W_i(y_{\mu+1})=(-\rho w_\mu)^{k_i}\left[\alpha_i+\sum_{l=1}^{m-1}\frac{\alpha_{i,\mu+1,l}}{\rho^l}+\frac{E}{\rho^m}\right]$$
$$+e^{-\rho w_\mu(b-a)}(-\rho w_\mu)^{k_i}\left[\beta_i+\sum_{l=1}^{m-1}\frac{\beta_{i,\mu+1,l}}{\rho^l}+\frac{E}{\rho^m}\right],$$

writing $w_{\mu+1}=-w_\mu$.

These expressions $W_i(y_j)$ are substituted in Δ and a factor

$$\prod_{i=1}^{n}\rho^{k_i}\cdot\prod_{j=\mu+2}^{n}e^{\rho w_j(b-a)}$$

is removed. The equation $\Delta=0$ becomes then of the form

$$(38)\quad \bar{\Delta} = \left(\theta_0 + \sum_{j=1}^{m-1} \frac{c_j}{\rho^j} + \frac{E_0}{\rho^m}\right) + e^{\rho w_\mu (b-a)} \left(\theta_1 + \sum_{j=1}^{m-1} \frac{d_j}{\rho^j} + \frac{E_1}{\rho^m}\right) + e^{-\rho w_\mu (b-a)} \left(\theta_2 + \sum_{j=1}^{m-1} \frac{e_j}{\rho^j} + \frac{E_2}{\rho^m}\right) = 0,$$

$$\Delta = \prod_{i=1}^{n} \rho^{k_i} \cdot \prod_{j=\mu+2}^{n} e^{\rho w_j (b-a)} \cdot \bar{\Delta},$$

θ_0, θ_1, θ_2 being defined as in (26). The equation (38) is quadratic in $e^{\rho w_\mu (b-a)}$.

This equation (38) will yield the forms stated in the theorem for the characteristic values λ_{II}, λ_{III} on S. They must lie near to one of the bounding rays, say $\arg \rho = (l_0 + 1)\pi/n$. If we consider the necessary form of the expansion on the other region S adjoining this ray, i. e., on the region

$$\frac{(l_0+1)\pi}{n} \leqq \arg \rho \leqq \frac{(l_0+2)\pi}{n}$$

we obtain the same equation (38) *except for different E-terms* perhaps; hence the same necessary form for λ_{II}, λ_{III} as before. Thus we have the necessary form for the entire λ-plane.

The final fact of unique existence is demonstrated as before unless

$$\text{either } \arg g_{II} = \begin{cases} 0 \\ \pi \end{cases} \qquad \text{or} \qquad \arg g_{III} = \begin{cases} 0 \\ \pi \end{cases}$$

when the values ρ_{II}, ρ_{III} may come indefinitely near to the bounding ray $\arg \rho = (l_0 + 1)\pi/n$. By considering first the case

$$p_2(x) = p_3(x) = \cdots = p_n(x) = 0,$$

when the solutions y_i are known, and by using continuity considerations, the fact of unique existence can be established for this case also.

§ 5. *Nature of the expansion.*

In this section we give a notion of the character of u_i, v_i and of the expansion (7).

Let us first develop the character of u_{II} in the case $n = 2\mu - 1$. Clearly

$$(39)\qquad u_{II} = c_1 y_1 + \cdots + c_n y_n$$

where we have for the determination of $c_1, c_2, \cdots, c_n$ the equations

$$(40)\qquad c_1 W_i(y_1) + c_2 W_i(y_2) + \cdots + c_n W_i(y_n) = 0 \quad (i = 1, 2, \cdots, n).$$

Substitute in the matrix $\| W_i(y_j) \|$ the expressions given in (31), (32), (33) and remove the factor ρ^{k_i} from the ith column $(i = 1, 2, \cdots, n)$; write for ρ the values (36). This matrix then takes the form

$$(41)\qquad \left\| \begin{array}{ccccccccccc} P_{11} & P_{12} & \cdots & P_{1\mu} & e^{l'\frac{w_{\mu+1}}{w_\mu}} P_{1,\mu+1} & \cdots & e^{l'\frac{w_n}{w_\mu}} P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2\mu} & e^{l'\frac{w_{\mu+1}}{w_\mu}} P_{2,\mu+1} & \cdots & e^{l'\frac{w_n}{w_\mu}} P_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ P_{n1} & P_{n2} & \cdots & P_{n\mu} & e^{l'\frac{w_{\mu+1}}{w_\mu}} P_{n,\mu+1} & \cdots & e^{l'\frac{w_n}{w_\mu}} P_{nn} \end{array} \right\|,$$

where the P_{ij} are of the form

$$(42)\qquad a+\frac{b}{l}+\cdots+\frac{E}{l^m},$$

and

$$l'=2l\pi\sqrt{-1}+\frac{2\pi\sqrt{-1}g_{II}}{n}.$$

The c_i are proportional to the minors of any row, say the first. The minors of the elements 1 to μ contain all the factors

$$\prod_{i=\mu+1}^{n} e^{l'\frac{w_i}{w_\mu}},$$

besides a factor like (42), while the minors of the element i, $i>\mu$, contain all these except $e^{l'w_i/w_\mu}$ besides a factor like (42). We conclude that

$$(43)\qquad \begin{aligned} c_i &= Q_i && (i=1,2,\cdots,\mu), \\ c_i &= e^{-l'\frac{w_i}{w_\mu}} Q_i && (i=\mu+1,\mu+2,\cdots,n), \end{aligned}$$

where Q_i is of the form (42).

If we substitute into (21) the values (36), we obtain

$$(44)\qquad y_i = e^{l'\frac{w_i}{w_\mu}\left(\frac{x-a}{b-a}\right)} Y_i \qquad (i=1,2,\cdots,n),$$

where Y_i is of the form

$$(45)\qquad a+\frac{b(x)}{l}+\cdots+\frac{E}{l^m}.$$

Placing the expressions for c_i and y_i of (43) and (44) in (39), we derive the form of u_{II}; likewise the form of u_{III}, v_{II}, v_{III} is obtained. We have finally

$$(46)\qquad \left\{ \begin{aligned} u_{II} &= \sum_{i=1}^{\mu-1} e^{l'\frac{w_i}{w_\mu}\left(\frac{x-a}{b-a}\right)} U_{Ii} + e^{l'\left(\frac{x-a}{b-a}\right)} U_{I\mu} + \sum_{i=\mu+1}^{n} e^{l'\frac{w_i}{w_\mu}\left(\frac{x-b}{b-a}\right)} \cdot U_{Ii}, \\ u_{III} &= \sum_{i=1}^{\mu-1} e^{l''\frac{w_i}{w_\mu}\left(\frac{x-b}{b-a}\right)} U_{IIi} + e^{l''\left(\frac{x-a}{b-a}\right)} U_{II\mu} + \sum_{i=\mu+1}^{n} e^{l''\frac{w_i}{w_\mu}\left(\frac{x-a}{b-a}\right)} U_{IIi}, \\ v_{II} &= \sum_{i=1}^{\mu-1} e^{-l'\frac{w_i}{w_\mu}\left(\frac{x-b}{b-a}\right)} V_{Ii} + e^{-l'\left(\frac{x-a}{b-a}\right)} V_{I\mu} + \sum_{i=\mu+1}^{n} e^{-l'\frac{w_i}{w_\mu}\left(\frac{x-a}{b-a}\right)} V_{Ii}, \\ v_{III} &= \sum_{i=1}^{\mu-1} e^{-l''\frac{w_i}{w_\mu}\left(\frac{x-a}{b-a}\right)} V_{IIi} + e^{-l''\left(\frac{x-a}{b-a}\right)} V_{II\mu} + \sum_{i=\mu+1}^{n} e^{-l''\frac{w_i}{w_\mu}\left(\frac{x-b}{b-a}\right)} V_{IIi}, \end{aligned} \right.$$

$$l' = 2l\pi\sqrt{-1} + \frac{2\pi\sqrt{-1}g_{II}}{n}, \qquad l'' = -2l\pi\sqrt{-1} - \frac{2\pi\sqrt{-1}g_{III}}{n}.$$

Here U, V are of the form (45) and $n = 2\mu - 1$.

When $n = 2\mu$ we get in the same manner

$$(47)\quad \begin{cases} u_{II} = \sum_{i=1}^{\mu-1} e^{l' \frac{w_i}{w_\mu}\left(\frac{x-a}{b-a}\right)} U_{Ii} + e^{l'\left(\frac{x-a}{b-a}\right)} U_{I\mu} + e^{-l'\left(\frac{x-a}{b-a}\right)} U_{I,\mu+1} + \sum_{i=\mu+2}^{n} e^{l' \frac{w_i}{w_\mu}\left(\frac{x-b}{b-a}\right)} U_{Ii}, \\ u_{III} = \sum_{i=1}^{\mu-1} e^{l'' \frac{w_i}{w_\mu}\left(\frac{x-a}{b-a}\right)} U_{IIi} + e^{l''\left(\frac{x-a}{b-a}\right)} U_{II\mu} + e^{-l''\left(\frac{x-a}{b-a}\right)} U_{II,\mu+1} + \sum_{i=\mu+2}^{n} e^{l'' \frac{w_i}{w_\mu}\left(\frac{x-b}{b-a}\right)} U_{IIi}, \\ v_{II} = \sum_{i=1}^{\mu-1} e^{-l' \frac{w_i}{w_\mu}\left(\frac{x-b}{b-a}\right)} V_{Ii} + e^{-l'\left(\frac{x-a}{b-a}\right)} V_{I\mu} + e^{l'\left(\frac{x-a}{b-a}\right)} V_{I,\mu+1} + \sum_{i=\mu+2}^{n} e^{-l' \frac{w_i}{w_\mu}\left(\frac{x-a}{b-a}\right)} V_{Ii}, \\ v_{III} = \sum_{i=1}^{\mu-1} e^{-l'' \frac{w_i}{w_\mu}\left(\frac{x-b}{b-a}\right)} V_{IIi} + e^{-l''\left(\frac{x-a}{b-a}\right)} V_{II\mu} + e^{l''\left(\frac{x-a}{b-a}\right)} V_{II,\mu+1} \\ \qquad\qquad + \sum_{i=\mu+2}^{n} e^{-l'' \frac{w_i}{w_\mu}\left(\frac{x-a}{b-a}\right)} V_{IIi}, \\ \qquad\qquad \cdot \end{cases}$$

$$l' = 2l\pi\sqrt{-1} + \frac{2\pi\sqrt{-1}g_{II}}{n}, \qquad l'' = 2l\pi\sqrt{-1} + \frac{2\pi\sqrt{-1}g_{III}}{n}.$$

In both (46) and (47) it is to be noted that the Σ-terms are not important except at $x = a$ and $x = b$, since the real parts of the exponential terms are large and negative for l large and $a < x < b$. We omit the fuller discussion, to be made by (46) and (47). It is worthy of remark that the simplest case

$$n = 1, \qquad L(u) = u', \qquad W_1(u) = u(a) - u(b)$$

yields the Fourier series. The general expansion is clearly of a similar nature.

§ 6. *Convergence of expansion to $f(x)$.*

We have expressed the sum of n terms of the expansion (7) in the form

$$I = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \int_a^b G(x, s; \lambda) f(s)\, ds\, d\lambda,$$

where Γ is some contour which contains $\lambda_1, \lambda_2, \cdots, \lambda_n$ but no other characteristic values within it; G is explicitly defined [see (10)] in terms of the solutions y_i of $L(u) + \lambda u = 0$. By means of the known asymptotic character of the y_i we determine the character of G, and then evaluate the contour integral as Γ enlarges without bound.

For convenience let us use the notation $[w]$ for an expression

$$w + \frac{a}{\rho} + \cdots + \frac{E}{\rho^m},$$

where $w, a, \cdots$ are independent of ρ but need not be constants.

THEOREM. *Let $f(x)$ be made up of a finite number of pieces in the interval $a \leqq x \leqq b$, each real, continuous, and with a continuous derivative. Unless $x = a$ or $x = b$, the expansion for $f(x)$ connected with the differential equation $L(u) + \lambda u = 0$ and the regular boundary conditions $W_i(u) = 0\ (i = 1, 2, \cdots, n)$ converges to $[f(x+0) + f(x-0)]/2$. At $x = a$ the series converges to*

$$a_1 f(a+0) + a_2 f(b-0)$$

and at $x = b$ to

$$b_1 f(a+0) + b_2 f(b-0),$$

*where a_1, a_2, b_1, b_2 are constants independent of $f(x)$.**

Proof. We restrict ourselves first to the case $n = 2\mu - 1$. At the end we outline a similar proof for $n = 2\mu$.

Let the contour Γ be taken as a circle $|\lambda| = k$ in the λ-plane. If the transformation $\lambda = \rho^n$ be made, we find

$$I = \int_{\gamma_1}\int_a^b n\rho^{n-1} G(x, s; \lambda) f(s)\, ds\, d\rho + \int_{\gamma_2}\int_a^b n\rho^{n-1} G(x, s; \lambda) f(s)\, ds\, d\rho.$$

In this expression γ_1 is the segment of a circle $|\rho| = k'$ lying on a region S,

$$S_1: \qquad \frac{l_0 \pi}{n} \leqq \arg \rho \leqq \frac{(l_0+1)\pi}{n},$$

and γ_2 is the segment of the same circle lying on the adjacent region S,

$$S_2: \qquad \frac{(l_0+1)\pi}{n} \leqq \arg \rho \leqq \frac{(l_0+2)\pi}{n}.$$

We confine our attention to the partial integral

$$I_{\gamma_1}^{a-x} = \int_{\gamma_1}\int_a^x n\rho^{n-1} G(x, s; \lambda) f(s)\, ds\, d\rho, \tag{48}$$

taking $a < x < b$. It is found to tend toward $f(x-0)/4$. By considerations of symmetry the limiting values of the remaining partial integrals are determined. The first part of the theorem then follows.

Let B be the bisection point of γ_1 and A, C its end points. Take AB as that segment of γ_1 for which

$$R(\rho w_\mu) < 0 \tag{49}$$

and BC as that segment of γ_1 for which

$$R(\rho w_\mu) > 0. \tag{50}$$

* For definitions of *regular conditions* see § 4. The restrictions imposed on $f(x)$ might be lightened but only at the cost of brevity.

The indices 1 to n of w_i are chosen as in (29) and (30). We must break up $I_{\gamma_1}^{a-x}$ in two parts I_{AE}^{a-x} and I_{EC}^{a-x}, both of which are attacked in the same way.

To treat G on AB, write

$$\bar{G}(x, s; \lambda) = \pm \tfrac{1}{2} \sum_{i=1}^{n} y_i(x)\bar{y}_i(s) \quad \begin{cases} + \text{ if } x > s, \\ - \text{ if } x < s, \end{cases} \tag{51}$$

where the y_i of (11) have been taken as the y_i of (21). Then $\bar{y}_i$ is the coefficient of τ_i in

$$\frac{\begin{vmatrix} \tau_1 & \tau_2 & \cdots & \tau_n \\ y_1^{(n-2)}(s) & y_2^{(n-2)}(s) & \cdots & y_n^{(n-2)}(s) \\ \cdot & \cdot & \cdots & \cdot \\ y_1(s) & y_2(s) & \cdots & y_n(s) \end{vmatrix}}{\begin{vmatrix} y_1^{(n-1)}(s) & y_2^{(n-1)}(s) & \cdots & y_n^{(n-1)}(s) \\ y_1^{(n-2)}(s) & y_2^{(n-2)}(s) & \cdots & y_n^{(n-2)}(s) \\ \cdot & \cdot & \cdots & \cdot \\ y_1(s) & y_2(s) & \cdots & y_n(s) \end{vmatrix}}.$$

But by (21)

$$y_i^{(j)}(s) = e^{\rho w_i(s-a)}\rho^j \left[w_i^j \right]. \tag{52}$$

If these expressions be substituted above we find

$$\bar{y}_i(s) = e^{-\rho w_i(s-a)} \frac{1}{n\rho^{n-1}} \left[-w_i \right]^* \tag{53}$$

since $-w_i/n$ is the coefficient of τ_i in the expression

$$\frac{\begin{vmatrix} \tau_1 & \tau_2 & \cdots & \tau_n \\ w_1^{n-2} & w_2^{n-2} & \cdots & w_n^{n-2} \\ \cdot & \cdot & \cdots & \cdot \\ 1 & 1 & \cdots & 1 \end{vmatrix}}{\begin{vmatrix} w_1^{n-1} & w_2^{n-1} & \cdots & w_n^{n-1} \\ w_1^{n-2} & w_2^{n-2} & \cdots & w_n^{n-2} \\ \cdot & \cdot & \cdots & \cdot \\ 1 & 1 & \cdots & 1 \end{vmatrix}}.$$

* The denominator contains the factors $e^{\rho w_i(s-a)}$ which we divide into the corresponding i-th columns of the numerator. It is to be remembered that, if $t \neq 0$,

$$\frac{1}{[t]} = \left[\frac{1}{t} \right].$$

In fact this coefficient m_i satisfies the conditions

$$\sum_{i=1}^{n} w_i^j m_i = 0 \quad (j=0, 1, \cdots, n-2) \qquad \sum_{i=1}^{n} w_i^{n-1} m_i = 1.$$

These show that m_i has the stated value.

By (51)

$$\overline{G}(x, s; \lambda) = +\tfrac{1}{2}\sum_{i=1}^{n} y_i(x)\overline{y}_i(s)$$

since $a < s < x$ for the partial integral I_{AB}^{a-x}. Also

$$W_i(\overline{G}) = -\tfrac{1}{2}\sum_{j=1}^{n} W_{ia}(y_j)\overline{y}_j(s) + \tfrac{1}{2}\sum_{j=1}^{n} W_{ib}(y_j)\overline{y}_j(s).$$

From these equations it is clear that if we multiply the columns 1 to μ of $N(x, s; \lambda)$ in (10) by $+\tfrac{1}{2}\overline{y}_i(s)$ $(i=1, 2, \cdots, \mu)$, the columns $\mu+1$ to n by $-\tfrac{1}{2}\overline{y}_i(s)$ $(i=\mu+1, \mu+2, \cdots, n)$ and add them to the last column, this last column has the elements

$$+\sum_{i=1}^{\mu} y_i(x)\overline{y}_i(s), \qquad \sum_{j=1}^{\mu} W_{ib}(y_j)\overline{y}_j(s) - \sum_{j=\mu+1}^{n} W_{ia}(y_j)\overline{y}_j(s) \quad (i=1, 2, \cdots, n).$$

By (21), (28), and (53) these are of the forms

$$-\frac{1}{n\rho^{n-1}}\sum_{i=1}^{\mu} e^{\rho w_i(x-s)}[w_i],$$

(54)

$$\frac{1}{n\rho^{n-1}}\left(\sum_{j=1}^{\mu} e^{\rho w_j(b-s)}\rho^{k_i}[-\beta_i w_j^{k_i+1}] + \sum_{j=\mu+1}^{n} e^{\rho w_j(a-s)}\rho^{k_i}[\alpha_i w_j^{k_i+1}]\right) \quad (i=1, 2, \cdots, n).$$

In the new notation equations (21), (31), (32), (33), and (34) give also

$$y_i = e^{\rho w_i(x-a)}[1] \qquad (i=1, 2, \cdots, n),$$

(55)

$$W_i(y_j) = \begin{cases} \rho^{k_i}[\alpha_i w_j^{k_i}] & (j=1, 2, \cdots, \mu-1), \\ \rho^{k_i}([\alpha_i w_\mu^{k_i}] + e^{\rho w_\mu(b-a)}[\beta_i w_\mu^{k_i}]) & (j=\mu), \\ \rho^{k_i} e^{\rho w_j(b-a)}[\beta_i w_j^{k_i}] & (j=\mu+1, \mu+2, \cdots, n), \end{cases}$$

$$\Delta(\lambda) = -\prod_{i=1}^{n}\rho^{k_i}\cdot\prod_{j=\mu+1}^{n} e^{\rho w_j(b-a)}\cdot([\theta_0] + e^{\rho w_\mu(b-a)}[\theta_1]).$$

We introduce these expressions into (10) where the elements of the last column of N have been modified to the form (54). The factors of the denominator $\Delta(\lambda)$ we can divide into the numerator as follows: the factor -1 into the last column, the factor ρ^{k_i} into the $(i+1)$-th row, the factor $e^{\rho w_j(b-a)}$ into the j-th column, the remaining factor $\overline{\Delta}$ into the μ-th column. We thus obtain

$$(56)\quad G(x, s; \lambda) = \frac{1}{n\rho^{n-1}} \times$$

$$\begin{vmatrix} e^{\rho w_1(x-a)}[1] \cdots & \dfrac{e^{\rho w_\mu(x-a)}[1]}{[\theta_0] + e^{\rho w_\mu(b-a)}[\theta_1]} & \cdots e^{\rho w_n(x-b)}[1] & \displaystyle\sum_{i=1}^{\mu} e^{\rho w_i(x-s)}[w_i] \\ [\alpha_1 w_1^{k_1}] \cdots & \dfrac{w_\mu^{k_1}([\alpha_1] + e^{\rho w_\mu(b-a)}[\beta_1])}{[\theta_0] + e^{\rho w_\mu(b-a)}[\theta_1]} & \cdots [\beta_1 w_n^{k_1}] & \displaystyle\sum_{j=1}^{\mu} e^{\rho w_j(b-s)}[\beta_1 w_j^{k_1+1}] \\ & & & \displaystyle -\sum_{j=\mu+1}^{n} e^{\rho w_j(a-s)}[\alpha_1 w_j^{k_1+1}] \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ [\alpha_n w_1^{k_n}] \cdots & \dfrac{w_\mu^{k_n}([\alpha_n] + e^{\rho w_\mu(b-a)}[\beta_n])}{[\theta_0] + e^{\rho w_\mu(b-a)}[\theta_1]} & \cdots [\beta_n w_n^{k_n}] & \displaystyle\sum_{j=1}^{\mu} e^{\rho w_j(b-s)}[\beta_n w_j^{k_n+1}] \\ & & & \displaystyle -\sum_{j=\mu+1}^{n} e^{\rho w_j(a-s)}[\alpha_n w_j^{k_n+1}] \end{vmatrix}.$$

From this form of G we can easily determine the limit of I_{AB}^{a-x}.

Assume the sequence of circles so chosen that their distance from the nearest point ρ_{II}, ρ_{III} is at least $d > 0$.* Then on AB when $|\rho|$ is large

$$(57)\qquad [\theta_0] + e^{\rho w_\mu(b-a)}[\theta_1] > M > 0.$$

In fact if

$$[\theta_0] + e^{\rho w_\mu(b-a)}[\theta_1] = \phi,$$

where $|\phi|$ is small, we infer at once that the point ρ lies near to ρ_{II} or ρ_{III}.

In view of (57), *every element of G given by* (56) *is bounded on AB* since the exponents have negative or zero real parts [see (29) and (49)]. Further, when this expression for G is used in I_{AB}^{a-x} the factor $n\rho^{n-1}$ cancels.

Consider those terms of I_{AB}^{a-x} which do not contain the first element of the last column; s is confined to this column. When we integrate these terms as to s, they have the form M/ρ (M bounded), as an integration by parts between the points of discontinuity of $f(s)$ shows. But these terms all contain as factor an element of the first row, not the last element, beside other bounded elements; since these first row terms are small for $|\rho|$ large, except the μ-th one when ρ is near to B,† we conclude that the ρ-integration of them will yield only small terms.‡

* This refers to the minimum distance of ρ_{II}, ρ_{III} from any point of the circle. In the proof of the theorem of § 4, ρ_{II}, ρ_{III} (corresponding to λ_{II}, λ_{III}) were found to be approximately equally spaced along the bisectors of S_1 and S_2. The above construction of a sequence of circles is therefore possible.

† Then ρw_μ is a pure imaginary.

‡ $$\left|\int_A^B \frac{\varepsilon}{\rho}\, d\rho\right| < \text{maximum of } \varepsilon \times \frac{\pi}{2n}.$$

This is small everywhere on AB if ε is small. Even if ε is small except for a little part of AB, this integral is small. 'Small' means of course indefinitely small as $|\rho|$ becomes indefinitely large.

Accordingly we need only consider the terms which contain the first element of the last column. This element has a coefficient 1, as it had originally in $G(x, s; \lambda)$. Here again we can restrict ourselves to terms in the brackets which do not contain $1/\rho$. Finally therefore

$$I_{AB}^{a-x} = \frac{1}{2\pi\sqrt{-1}} \int_A^B \int_a^x \left(\sum_{i=1}^{\mu} e^{\rho w_i (x-s)} w_i \right) f(s)\, ds\, d\rho + \epsilon,$$

where ϵ signifies a quantity which tends to 0 as $|\rho|$ increases.

Integrating by parts we find

$$\int_a^x e^{\rho w_i (x-s)} w_i f(s)\, ds = \frac{f(x-0)}{\rho} + \frac{\epsilon}{\rho} \qquad (i = 1, 2, \cdots, \mu),$$

where ϵ is certainly small unless $i = \mu$ and ρ is near to B. From this we conclude that

$$\int_A^B \int_a^x e^{\rho w_i (x-s)} w_i f(s)\, ds\, d\rho = \frac{\pi\sqrt{-1}}{2n} f(x-0) + \epsilon,$$

and therefore

$$I_{AB}^{a-x} = \frac{\mu}{4n} f(x-0) + \epsilon. \tag{58}$$

To treat I_{BC}^{a-x} we multiply the columns 1 to $\mu - 1$ of $N(x, s; \lambda)$ by $+\frac{1}{2}\bar{y}_i(s)$, and the columns μ to n by $-\frac{1}{2}\bar{y}_i(s)$ and add them to the last column. The elements of the μ-th column can be written in this case

$$\frac{e^{\rho w_\mu (x-b)}}{e^{\rho w_\mu (a-b)}[\theta_0] + [\theta_1]}, \qquad \frac{w_\mu^{k_i}\left(e^{\rho w_\mu (a-b)}[\alpha_i] + [\beta_i]\right)}{e^{\rho w_\mu (a-b)}[\theta_0] + [\theta_1]} \qquad (i = 1, 2, \cdots, n),$$

and all the elements are bounded as before. The important term again comes from the first element of the last column, and we find precisely as before that

$$I_{BC}^{a-x} = \frac{\mu - 1}{4n} f(x-0) + \epsilon. \tag{59}$$

Adding (58) and (59) we find

$$I_{\gamma_1}^{a-x} = \tfrac{1}{4} f(x-0) + \epsilon,$$

and by symmetry

$$I_{\gamma_2}^{a-x} = \tfrac{1}{4} f(x-0) + \epsilon, \qquad I_{\gamma_1}^{x-b} = \tfrac{1}{4} f(x+0) + \epsilon, \qquad I_{\gamma_2}^{x-b} = \tfrac{1}{4} f(x+0) + \epsilon.$$

Hence we find

$$I_{\Gamma}^{a-b} = \frac{f(x-0) + f(x+0)}{2} + \epsilon.$$

This proves the first part of the theorem.

When $x = b$ the preceding work must be modified, since now the elements

$\mu + 1$ to n of the first row in (56) have the form $[1]$. Other elements of the last column now become important beside the first. An integration by parts with respect to s followed by a ρ-integration gives us terms

$$b_1 f(a + 0) + b_2 f(b - 0).$$

The same result must of course hold at $x = a$.

When $n = 2\mu$ the attack is almost the same. One obtains a determinant expression for G by modifying the last column and distributing the two factors of $\bar{\Delta}$ into the μ-th and $(\mu + 1)$-th column.* The elements are bounded on γ_1 in this case and one obtains first $I_{\gamma_1}^{a-x}$ which tends toward $\frac{1}{2} f(x - 0)$. The remainder of the proof is as before.

* See (38) in which $\bar{\Delta}$ is factorable into

$$([\phi_1] + e^{\rho w_\mu (b-a)}[\psi_1])([\phi_2] + e^{-\rho w_\mu (b-a)}[\psi_2]).$$

AN APPLICATION OF THE FORM-PROBLEMS ASSOCIATED WITH CERTAIN CREMONA GROUPS TO THE SOLUTION OF EQUATIONS OF HIGHER DEGREE*

BY

ARTHUR B. COBLE

Introduction.

Certain groups of Cremona transformations isomorphic with the symmetric group of order $n!$ are discussed in this paper. They are seen to define form-problems in much the same way as groups of collineations, the known quantities or parameters in the problem being the invariants of the n-ic under linear transformation. The well-known theory of binary forms is thus utilized to the utmost in reducing the number of parameters that occur in the general equation of degree n.

The earlier paragraphs are devoted to a detailed study of the quintic. These have close contact at many points with KLEIN's *Ikosaeder*. In § 1 and § 2 Cremona groups of order 120 and their invariants are given. In § 3 the form-problems determined by these groups are formulated. They contain as parameters two absolute invariants of the quintic. For want of a more direct method,† in § 4 the Cremona form-problem is made to depend on KLEIN's "problem of the A's." It is probable that this form of the reduction of the quintic to the A-problem is as simple as can be devised.‡ It is indeterminate only so far as the choice of the two absolute invariants is concerned. Since the solution of the A-problem proposed in the *Ikosaeder* is not carried out completely, a new solution is given in § 5 and all the necessary calculations are made. The summary in § 6 shows that the solution of the quintic, after the adjunction of the square root of its discriminant, is obtained by an extraction of a square root, a determination of the ikosahedral irrationality and furthermore by substitutions in explicitly given rational expressions.

The discussion of the general case in § 8 emphasizes the importance of this

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† See the opening remarks of § 7.

‡ See other methods suggested in the *Ikosaeder*, p. 248, § 6.

novel Cremona form-problem, both as effecting an immediate reduction in the number of parameters found in the original problem, and as being the first in a series of successively simpler form-problems by means of which the given equation can be solved.

The distinction preserved by KLEIN between the "form-problem" and its accompanying "equation-problem" is here not maintained. In fact, nearly all the problems referred to hereafter as form-problems are, in reality, equation-problems. The two are very closely connected, however, and the former term is much more characteristic. Strictly speaking, a Cremona group cannot define a form-problem.

§ 1. *The Cremona group of order* 120 *determined by five points in a plane.**

The group is suggested by this problem: *Given two sets of n points,* $p_0, p_1, \cdots, p_{n-1}$ *and* $q_0, q_1, \cdots, q_{n-1}$, *in the same or different planes, the two sets being ordered with regard to each other, under what conditions can points* x *and* y *be found such that the pencil of lines from* x *to the points* p_i *is projective to the pencil of lines from* y *to the points* q_i*?*

When n is 3, there is no limitation on x and y except that if x is on a line $\overline{p_i p_k}$, y is on the line $\overline{q_i q_k}$.

When n is 4, if x is any point on a conic through $p_0, \cdots, p_3$, y is any point on a definite conic through $q_0, \cdots, q_3$. A one-to-one relation is thus established between the pencils of conics through the two sets of four points, which is fixed by the requirement that the three line pairs of the one pencil correspond to the three line pairs of the other.

When n is 5, there is a one-to-one correspondence between the points x and y. They are corresponding points in a Cremona transformation, T. For if x is given, to the conics $\overline{xp_0p_1p_2p_3}^2$ and $\overline{xp_0p_1p_2p_4}^2$ correspond conics $\overline{yq_0q_1q_2q_3}^2$ and $\overline{yq_0q_1q_2q_4}^2$ which meet at q_0, q_1, q_2 and the required fourth point y. To determine the order of T we examine its principal points. If x is at p_0, it determines only a four-line pencil $p_0 - p_1, p_2, p_3, p_4$. Hence y is any point on a conic through q_1, q_2, q_3, q_4. The five points p_i are, then, double principal points of T and the five points q_i are double principal points of T^{-1}. If x is any point on the conic $\overline{p_0p_1p_2p_3p_4}^2$, y is a definite point q. Therefore q is a sixth double principal point of T^{-1}. If y is any point on the conic $\overline{q_0 \cdots q_4}^2$, x is a definite point p, the sixth double principal point of T. For all other points the transformation is determinate, i. e., T *is a Cremona transformation of the fifth order with six double principal points.*

* The Cremona G_{120} of § 2 has been studied exhaustively by H. E. SLAUGHT, American Journal of Mathematics, vol. 22 (1900), p. 343 and vol. 23 (1901), p. 99. In the later paper the connection between the invariants of the group and the invariants of the quintic is obtained by a method different from that of § 2. The generalized group, $G_{n!}$ in S_{n-3}, considered in § 8 is due to E. H. MOORE, American Journal of Mathematics, vol. 22 (1900), p. 279.

Moreover T is the most general Cremona transformation of that type. If $r_0, r_1, \cdots, r_5$ and $s_0, s_1, \cdots, s_5$ are the two sets of double principal points of such a transformation, r_i and s_i being "opposite" points, i. e., points such that the principal conic of the one does not pass through the other, the pencil of conics through four points r_i corresponds to the pencil of conics through the four opposite points s_i in such a way that the line pairs in the two pencils correspond. Hence if x and y are a pair of corresponding points, the six lines from x to r_i are projective to the six lines from y to s_i. If x is at r_i, y is any point on the conic through the five points q_k, $k \neq i$, and the five lines from r_i to r_k are projective to the five points q_k on their conic. Two sets of six points with this property will be called a "double-six-point." *

The necessary and sufficient condition that two sets of six points be the double principal points of a Cremona transformation, T, of the fifth order is that the two sets form a double-six-point. If x and y are a pair of corresponding points the sextic line pencils from x and y, to the principal points of T and T^{-1} respectively, are projective.

T is fully defined when five pairs of opposite points are given. If the five points q coincide with the five points p, q_i falling on p_i, the transformation is the identity. And if the points q, in some permutation, fall on the points p, the transformation is of the fifth order and periodic, with the same period as the permutation. By employing the 120 permutations of the points q_i with reference to a fixed order of the points p_i, a group of 120 transformations is obtained isomorphic with the symmetric group on five things. Any transformation can be represented by a permutation. Thus $\left(\begin{smallmatrix} 0 & 1 & 2 & 3 & 4 \\ a_0 & a_1 & a_2 & a_3 & a_4 \end{smallmatrix}\right)$ is the transformation in which the principal conic of p_0 passes through $p_{a_1}, p_{a_2}, p_{a_3}, p_{a_4}$, and through the sixth principal point of T^{-1}, which will be denoted by p with a subscript giving the inverse permutation in cycle form. A set of 120 conjugate points determine 120 five-line pencils to the points p_i, which are all projective in various orders. Thus a binary quintic determines a set of 120 conjugate points in the plane and a set of 120 conjugate points determines a system of ∞^3 projective binary quintics. This representation is determinate except for points on the conic $\overline{p_0 \cdots p_4}^2$. All these points together with the 119 sixth principal points represent the same system of projective quintics. Some very degenerate quintics also are not represented by any points, e. g., quintics with a triple root.

* The double-six-point occurs in a number of connections: If a cubic surface be represented on a plane by means of six skew lines of the surface, the two sets of six points, arising from the two representations by means of the two opposite sets of lines in a double six, form a double-six-point; the six pairs of points, which satisfy four linearly independent bilinear relations, form a double six-point; the six pairs of corresponding points common to two quadratic transformations form a double-six-point; in a Cremona transformation of the fourth order with one triple and six simple principal points, the two sets of six points form a double-six-point and, with reference to them, the two triple points are corresponding; the six elements common to four connexes (1, 1), as well as the six polar points and lines common to two triangles, form a double-six-point-line, the half-dual of the double-six-point.

It is clear that the locus of points x, for which an invariant of the five-line-pencil vanishes, is an invariant curve of the group. That the converse is true will be shown in the next paragraph. Hence,

There is a one-to-one correspondence between the invariant curves of the Cremona G_{120} and the invariants of the binary quintic.

These invariant loci have been derived by CLEBSCH [Mathematische Annalen, vol. 4 (1871)] by means of his principle of transference, in the dual case of five lines. However, the existence of the underlying Cremona group is not pointed out. The properties of the various loci are treated in some detail. By the use of the group some of these properties are rendered very evident. In what follows, the notation of CLEBSCH (*Binäre Formen*) for the comitants of the quintic is used.

A transformation T of the fifth order has seven fixed points or a locus of fixed points. If T is involutory it has necessarily a locus of fixed points of odd order. Considering in particular $T_{(01)}$ we see that every point on the line $\overline{p_0 p_1}$ is a fixed point. Each conic in the pencil of conics through $p_2, p_3, p_4, p_{(01)}$ is unaltered. Each conic cuts the line $\overline{p_0 p_1}$ in two points, the fixed points of the involution on that conic. The two conics through p_2, p_3, p_4, touching $\overline{p_0 p_1}$ at p_0 and at p_1 respectively are the principal conics of p_0 and p_1 and meet at $p_{(01)}$. The only fixed points other than points on $\overline{p_0 p_1}$ are the vertices of the diagonal triangle of $p_2, p_3, p_4, p_{(01)}$.

By the transformation $T_{(01)(23)}$, a cubic curve with a double point at p_4 and simple points at p_0, p_1, p_2, p_3 is transformed into another of the same kind. Of the linear system containing the ∞^2 such curves, a pencil is unaltered, member by member, also one other cubic, $C_{(01)(23)}$. Of the points adjacent to p_4, those two on the line pair apolar to the pairs $\overline{p_4 p_0}, \overline{p_4 p_1}$ and $\overline{p_4 p_2}, \overline{p_4 p_3}$ are fixed. They determine the tangents to $C_{(01)(23)}$ at p_4. The pairs of tangents apolar to this pair belong to the members of the invariant pencil. Each member cuts $C_{(01)(23)}$ in a fixed point. The ninth base point of the pencil is also a fixed point. The two degenerate members of the pencil $\overline{p_4 p_0}\ \overline{p_4 p_1}\ \overline{p_2 p_3}$ and $\overline{p_4 p_2}\ \overline{p_4 p_3}\ \overline{p_0 p_1}$ show this point to be the meet of $\overline{p_2 p_3}$ and $\overline{p_0 p_1}$, say $P_{01,23}$. The other two diagonal points $P_{02,31}$ and $P_{03,12}$ lie on $C_{(01)(23)}$. In the invariant pencil of cubics there is only one which breaks up into a line and a conic, $\overline{p_0 p_1 p_2 p_3 p_4^2}\ \overline{p_4 P_{01,23}}$. The conic, being a principal conic, has no transform, and the transform of $\overline{p_4 P_{01,23}}$ must be the line itself and the conic $\overline{p_0 \cdots p_4^2}$. Hence the sixth principal point, $p_{(10)(23)}$, lies on the line $\overline{p_4 P_{01,23}}$. On this line is an involution whose double points are $P_{01,23}$ and the third meet of the line with $C_{(01)(23)}$. Hence $p_{(01)(23)}$ must coincide with p_4 along the line $\overline{p_4 P_{01,23}}$. Otherwise it would have two correspondents, the point p_4 and the point where the line meets the conic $\overline{p_0 \cdots p_4^2}$. Each conic of the pencil through $p_0 \cdots p_3$ is unaltered, the double points of the involution on any one being the two remaining meets with $C_{(01)(23)}$.

The two types considered above include all involutory Cremona transformations of the fifth order with six double principal points.

The transformation $T_{(012)}$ has seven fixed points, but three are accounted for by the fact that three principal conics pass through their principal points. The line $\overline{p_3 p_4}$ is unaltered and the collineation upon it has two fixed points, the Hessian pair of the three meets with the sides of the triangle p_0, p_1, p_2. The other two fixed points are the two meets of the three cubics, $C_{(12)(34)}$, $C_{(20)(34)}$ and $C_{(01)(34)}$. These two sets of two points occur in two different sets of 20 conjugate points. The points of the first kind are invariant under the cyclic $G_6, (012)(34)$, and are, then, the fixed points of the transformations of period six. Those of the second kind are invariant under a dihedral G_6 generated by (012) and $(12)(34)$. To points of the first kind correspond quintics made up of a cubic and the square of one of its Hessian points; to those of the second kind correspond quintics made up of a cubic and its Hessian.

The transformation $T_{(0123)}$ has three fixed points. It interchanges the lines $\overline{p_0 p_2}$ and $\overline{p_1 p_3}$, and therefore $P_{02,13}$ is a fixed point. The conics of the pencil on p_0, p_1, p_2, p_3 are interchanged involutorily. The line-pair $\overline{p_0 p_2}, \overline{p_1 p_3}$ is a fixed conic and the other two line pairs are interchanged. Hence the other fixed conic is the polar of the first line pair as to the other two (in the sense of polarity in a linear system) and must be one of the three harmonic conics on the four points. The cubic $C_{(02)(31)}$ is also unchanged. The two remaining fixed points are the two meets of this cubic and the harmonic conic. The corresponding quintic is a cyclo-projective set of four points and a double point of the cycle.

Finally, the transformation $T_{(01234)}$ has two fixed points which are the points common to the cubics $C_{(14)(23)}$, $C_{(20)(34)}$, $C_{(31)(40)}$, $C_{(42)(01)}$, $C_{(03)(12)}$. Each point is unaltered by a dihedral G_{10} and represents a cyclo-projective quintic.

Summing up the above, *we find eight sets of conjugate points which number less than* 120.

(1) Sets of 60 conjugate points arising from a general point on the locus, $\sqrt{D} = \Pi(\overline{p_i p_k})$. The corresponding quintics are of the type $z_1^2 z_2 (z_1 + z_2)(z_1 + \alpha z_2)$, and the corresponding subgroups of the type (01).

(2) Sets of 60 conjugate points arising from a general point on the locus, $R = \Pi(C_{(ik)(lm)})$. The quintics are of the type $z_1(z_1^2 + z_2^2)(z_1^2 + \alpha z_2^2)$, and the subgroups of the type (01)(23).

(3) A set of 30 conjugate points Q. These are the diagonal points of the ten 4-points, $p_{(ik)}, p_l, p_m, p_n$. On a line $\overline{p_i p_k}$, they are the cubic covariant points of the cubic cut out by the sides of the triangle p_l, p_m, p_n. The quintics are of the type $z_1^2 z_2 (z_1^2 - z_2^2)$ and the subgroups of the type (01), (23), (01)(23).

(4) A set of 30 conjugate points Q'. They arise from the two meets of a cubic $C_{(01)(23)}$ with a definite harmonic conic on $p_0, \cdots, p_3$. The quintics are of the type $z_1(z_1^4 - z_2^4)$ and the subgroups of the type (0123).

(5) A set of 20 conjugate points H. They are the Hessian points on the lines $\overline{p_i p_k}$. The quintics are of the type $z_1^2(z_1^3 - z_2^3)$ and the subgroups of the type (012)(34).

(6) A set of 20 conjugate points I. They are the meets of the cubics $C_{(ik)(lm)}$ by threes. The quintics are of the type $z_1 z_2(z_1^3 - z_2^3)$ and the subgroups of the type (012), (12)(34).

(7) A set of 15 conjugate points $P_{01,23}$. They are the fifteen diagonal points of the five points taken by fours. The quintics are of the type $z_1^2 z_2^2(z_1 + z_2)$, and the subgroups of the type (0231), (03).

(8) A set of 12 conjugate points K. They are the meets of the cubics $C_{(ik)(lm)}$ by fives. The quintics are of the type $z_1^5 - z_2^5$, and the subgroups of the type (01234), (14)(23).

That this enumeration is complete follows from the fact that there are no other types of binary quintics which admit a linear transformation. The points I separate into two sets of 10, each set conjugate under the alternating G_{60}. Under the same group the 12 points K separate into two sets of 6 conjugate points. All the sets of points found above are referred to in the quoted article of CLEBSCH except the set (3). He finds that the invariant curves A, B, C, R are of orders 10, 20, 30, 45 respectively with 4-, 8-, 12-, 18-fold points respectively at $p_0, \cdots, p_4$. The curves R and $D = A^2 - 64B$ have already been noted. Since the pencil $A^2 + \lambda B$ contains two perfect squares each curve of order 20 breaks up into two curves of order 10, each invariant under G_{60}.

It is well known that the invariants B and C vanish if the quintic is $z_1^5 - z_2^5$. Hence the curves B and C pass through the 12 points K. CLEBSCH shows that, at each K-point, B has a double point and C a double cusp whose cusp tangents are the tangents of the double point of B. This fact, important in what follows, is easy to prove by the group properties. Let K be a definite K-point and B_1 the factor of B which passes through it. K is a fixed point of a dihedral G_{10}. Hence the directions about K divide into a set of two, two sets of five, and ∞^1 sets of 10 conjugate directions. An invariant curve through K must have there two, five or ten tangents. B_1, a curve of order 10 with five 4-fold points at p_i, cannot have at K a point of order 4 or more, else it would have six such points and vanish identically. Therefore it must pass through K with the definite set of two tangents. B_1 and C meet in $30 \cdot 10 - 5 \cdot 4 \cdot 12 = 60$ points outside the points p_i. This set of 60 points is conjugate under G_{60}. C has at K a double point or a double cusp with the same tangents as B_1. If it has only a double point, $6 \cdot 6$ of the 60 points are accounted for. But the remaining 24 would also be at the 6 K-points, for they only can form a set of 24 conjugate points. Also C can not have five-fold points at the 6 K-points. Else it would meet one of the 15 cubics $C_{(ik)(lm)}$ in $12 \cdot 6 + 4 \cdot 5 = 92$ points and contain it as a factor, but this is impossible. C must therefore have at K a double cusp with the same tangents as B_1.

CLEBSCH also shows that, outside the points p_i, the curves A and B meet in the points H and the curves A and C touch at the 30 points Q.

There are two objections to the present form of the group. One is the very high order of the invariant curves. The second — more weighty with regard to the application of the group in the solution of the quintic — is that one system of projective binary quintics is represented by all the points on a conic. If the group be transformed by a Cremona transformation, with this conic as a principal conic, introducing thus no new principal points, this objection will be removed. Such a transformation, S, is the cubic involution with one of the five points, say p_0, as a double point and the other four as simple principal points. S is the projection of each conic of the pencil through $p_1, \cdots, p_4$ into itself from p_0. The transformed group will be discussed in the next paragraph.

§ 2. *The Cremona group of order* 120 *determined by four points in a plane.*

The transformed group is obtained most simply by a construction for the set of 120 conjugate points determined by a binary quintic. If x is one of the points in question, the five lines $\overline{xp_i}$ must be projective to the given quintic. These five lines are projective also to the five tangents at x to the five conics through x and four each of the points p_i. For x and p_i determine a pencil of cubics with a double point at x. The tangents to these cubics are pairs of an involution which includes, for the degenerate members, the five pairs mentioned. If S transforms x into y, the five conics become the conic $\overline{yp_1p_2p_3p_4}$ and the four lines $\overline{yp_1}, \cdots, \overline{yp_4}$. Naming the points by the subscripts only, we find therefore

(1) *In the new form of the group a point y represents the system of quintics projective to the four lines $\overline{y1}, \cdots, \overline{y4}$ and the tangent at y to the conic $y1234^2$, referred to hereafter as yT.*

The group is determined by four points only. Its properties can be deduced from those of the first group or discussed analytically with ease. Let the points 1, 2, 3, 4 have respectively the coördinates 1, −1, −1; −1, 1, −1; −1, −1, 1; 1, 1, 1 and let

$$\begin{aligned} s_1 &= y_1 - y_2 - y_3, & s_2 &= -y_1 + y_2 - y_3, \\ s_3 &= -y_1 - y_2 + y_3, & s_4 &= y_1 + y_2 + y_3, \end{aligned} \tag{2}$$

$$s_1 + s_2 + s_3 + s_4 = 0.$$

(3) *The binary value system $\infty, s_1, s_2, s_3, s_4$ is projective to the five lines $yT, \overline{y1}, \overline{y2}, \overline{y3}, \overline{y4}$.*

This statement is verified at once by calculating two independent anharmonic ratios. If one root of a quintic is infinite and the sum of the other four zero, the four finite roots are determined projectively to within a factor of propor-

tionality. We find that, if the roots are $r_0, \cdots, r_4$, and r_0 is transformed to infinity and the sum of the remaining transformed roots made zero, to within a factor these are

$$(4)\quad \begin{aligned} s_1 &= (12)(30)(40) + (13)(40)(20) + (14)(20)(30), \\ s_2 &= (21)(30)(40) + (23)(40)(10) + (24)(10)(30), \\ s_3 &= (31)(40)(20) + (32)(40)(10) + (34)(10)(20), \\ s_4 &= (41)(20)(30) + (42)(30)(10) + (43)(10)(20) \quad [(ik) = r_i - r_k]. \end{aligned}$$

Therefore

$$(5)\quad \begin{aligned} &s_4 - s_1 = 2(y_2+y_3) = 4\ (41)(20)(30), \quad s_2 - s_3 = 2(y_2-y_3) = 4\cdot(23)(10)(40), \\ &s_4 - s_2 = 2(y_3+y_1) = 4\cdot(42)(30)(10), \quad s_3 - s_1 = 2(y_3-y_1) = 4\cdot(31)(20)(40), \\ &s_4 - s_3 = 2(y_1+y_2) = 4\cdot(43)(10)(20), \quad s_1 - s_2 = 2(y_1-y_2) = 4\cdot(12)(30)(40) \end{aligned}$$

$$(6)\quad \begin{aligned} y_1 &= (43)(10)(20) + (12)(30)(40) = (42)(30)(10) - (31)(20)(40), \\ y_2 &= (41)(20)(30) + (23)(10)(40) = (43)(10)(20) - (12)(30)(40), \\ y_3 &= (42)(30)(10) + (31)(20)(40) = (41)(20)(30) - (23)(10)(40). \end{aligned}$$

Formulas (2) and (3) give one binary quintic determined by the point y and formulas (6) the point y determined by a given binary quintic.

To obtain the transformations of the group, we have merely to permute the differences of the roots in the required way and express the coördinates of the new point y' in terms of those of the old by eliminating the differences. First of all it is clear that any permutation of the roots $r_1, \cdots, r_4$ alone determines the same permutation of $s_1, \cdots, s_4$; i. e., the group contains the collineation group of the 4-line $s_i = 0$ or, also, of the 4-point 1, 2, 3, 4. To get the transformation $T_{(04)}$ we have

$$(7)\quad \begin{aligned} y_2' + y_3' &= \rho(01)(24)(34) = \sigma(01)(02)(03)\cdot(01)(24)(34) = \sigma(y_3+y_1)(y_1+y_2), \\ y_2' - y_3' &= \rho(23)(14)(04) = \sigma(01)(02)(03)\cdot(23)(14)(04) = \sigma(y_3-y_2)(y_2+y_3), \\ y_1' + y_2' &= \rho(03)(14)(24) = \sigma(01)(02)(03)\cdot(03)(14)(24) = \sigma(y_3+y_1)(y_2+y_3), \end{aligned}$$

a quadratic involution with the fixed point 4 and principal triangle 123. The line $\overline{31}$ corresponds to the point 2 and the line 23 to the point 1, i. e., the involution is non-inversive.

(8) *The group is generated by the collineation group G_{24} of the four points* 1, 2, 3, 4 *and a quadratic non-inversive involution with one of the points fixed and the other three as principal points.*

It is made up of the G_{24} and 96 quadratic transformations with principal points at the four points. The five pencils of conics through 4 of the 5 points

p_i are in this case the pencil of conics through 1, 2, 3, 4, and the four pencils of lines on the points i. The invariant curve $\sqrt{D}$ is the six lines through the four points. The fifteen cubics $C_{(ik)(lm)}$ become three lines and twelve conics. Since they are the loci of fixed points of the even involutions we find the three lines from G_{24}. They are $y_i = 0$, the axes of the harmonic perspectivities with centers $u_i = 0$ respectively, and correspond to the permutations $(4i)(kl)$ $i, k, l = 1, 2, 3$. These determine simply the change of sign of y_k and y_l and with the identity form the invariant subgroup G_4 of G_{24}. If these perspectivities are transformed by $T_{(04)}$, (7), we find the three inversions in a conic which correspond to the permutations $(0i)(kl)$. Thus for the permutations $(01)(23)$, $(02)(31)$ and $(03)(12)$ respectively the conics are

$$
\begin{aligned}
y_1^2 - y_2^2 - y_3^2 - (y_2y_3 + y_3y_1 + y_1y_2) &= 0, \\
-y_1^2 + y_2^2 - y_3^2 - (y_2y_3 + y_3y_1 + y_1y_2) &= 0, \\
-y_1^2 - y_2^2 + y_3^2 - (y_2y_3 + y_3y_1 + y_1y_2) &= 0.
\end{aligned} \tag{9}
$$

The center of inversion for each is the point 4, but the axes of inversion are $y_2 + y_3$, $y_3 + y_1$, $y_1 + y_2$, respectively. The other 9 conics of inversion are found from these three by transformation with G_4.

In giving the sets of conjugate points less than 120 in number, those obtained from one by the operations of G_{24} are not written. It is clear also that points on a line $\overline{p_0p_i}$ in § 1 are in this case directions about i. Taking up the points in the same order as before we find:

(3) The 30 points Q include the directions at 4 on the three cubic covariant lines of $4i$. These directions are on the lines $-2y_1 + y_2 + y_3$, $y_1 - 2y_2 + y_3$, $y_1 + y_2 - 2y_3$. Their transforms give twelve points like 2, 1, 1, and six like 0, 1, 2.

(4) The 30 points Q' are obtained by taking the intersections of the harmonic conic $-2y_1^2 + y_2^2 + y_3^2 = 0$ and the axis $y_1 = 0$, each determined by the cyclic $G_4(1243)$. They include the six points like 0, 1, i and their transforms like $1, 1 + 2i, 1 - 2i$.

(5) The 20 points H include eight directions, the two at 4 being on the lines $y_1 + \omega y_2 + \omega^2 y_3 = 0$, $y_1 + \omega^2 y_2 + \omega y_3 = 0$ $(\omega = e^{2\pi i/3})$; their transforms are the 12 points like $\omega - \omega^2$, 1, -1.

(6) The 20 points I are determined by taking the intersections of properly chosen factors of R, e. g., $y_1 = 0$ and the first conic (9). They are the transforms of the pairs $\left\{ \begin{smallmatrix} 0, & \omega, & -\omega^2 \\ 0, & \omega^2, & -\omega \end{smallmatrix} \right\}$, $\left\{ \begin{smallmatrix} 1, & \omega, & \omega^2 \\ 1, & \omega^2, & \omega \end{smallmatrix} \right\}$, by the even G_{12}. The points in the upper, and in the lower, row, form the two sets of conjugate points under G_{60}.

(7) The 15 points $P_{ik,\,lm}$ are the three diagonal points of reference and the twelve directions at the four points on the six lines joining them.

(8) The 12 points K are the meets of properly chosen factors of R, e. g.,

y_1 and the second conic in (9). They are the transforms of the pair $\left\{\begin{smallmatrix}0, & 1, & \delta_1\\ 0, & 1, & \delta_2\end{smallmatrix}\right\}$ by G_{12}, the points in the upper and lower rows forming two sets of 6 conjugate points under G_{60}. Here $\delta_1 = \epsilon + \epsilon^4$, $\delta_2 = \epsilon^2 + \epsilon^3$, $\epsilon = e^{2\pi i/5}$.

The invariant curves A, B, C, R, have now the orders 6, 12, 18, 27, respectively and 2-, 4-, 6-, 9-fold points respectively at the points i. Their equations are easily deduced from a typical representation of the binary quintic given in CLEBSCH's *Binäre Formen* (p. 353). The representation is as follows:

If $F(z) = a_z^4 = b_z^4 = \cdots$ be a binary quartic, x any point *not a root of the quartic* and $a_x^3 a_z$ its linear polar as to F, x and its polar are taken as new reference points. The transformation is

$$\eta = (xz) = \eta_1 z_1 + \eta_2 z_2, \qquad \xi = a_x^3 a_z = \xi_1 z_1 + \xi_2 z_2.$$

Since $a_z(\xi\eta) \equiv (a\eta)\xi - (a\xi)\eta$ we have, on raising to the fourth power,

$$(\xi\eta)^4 \cdot F(z) = (a\eta)^4 \cdot \xi^4 - 4(a\eta)^3(a\xi)\cdot\xi^3\eta + 6(a\eta)^2(a\xi)^2\cdot\xi^2\eta^2 - 4(a\eta)(a\xi)^3\cdot\xi\eta^3 + (a\xi)^4\cdot\eta^4.$$

Here the coefficients $(\xi\eta)^4$ and $(a\eta)^i(a\xi)^{4-i}$ are invariants of F and η, i. e., covariants of F in the variable x. Evaluating the covariants in terms of the complete system and dividing out by $F(x)$, by hypothesis not zero, we have the typical representation:

$$F^3 \cdot F(z) = \xi^4 + 3H\xi^2\eta^2 + 4T\xi\eta^3 + \tfrac{1}{2}iF^2 - \tfrac{3}{4}H^2\eta^4. \tag{10}$$

Since η or x was any point, $\eta \cdot F(z)$ is a general quintic with a system of coefficients

$$0,\ 1,\ 0,\ \frac{3H}{2},\ 4T,\ 5(\tfrac{1}{2}iF^2 - \tfrac{3}{4}H^2). \tag{11}$$

CLEBSCH finds the invariants of this quintic to be

$$\begin{aligned}
A &= -48F^2(4iH - jF),\\
B &= 2F^4\{18(4iH - jF^2) - 125F^2(i^3 - 6j^2)\},\\
C &= 8F^6\{H^3(8\cdot 48i^3 + 24\cdot 25\cdot 48j^2) - H^2F\cdot 48\cdot 81i^2j\\
&\qquad - HF^2(7\cdot 48\cdot 48ij^2 + 25\cdot 15i^4/2) + F^3(25\cdot 80i^3j + 8\cdot 24\cdot 32j^3)\},\\
R &= 12\cdot 250TF^9\{-H^3\cdot 48\cdot 48(48j^3 + 14i^3j) + 6\cdot 48H^2F(24i^2j^2 + 7i^5)\\
&\qquad - 48\cdot 243HF^2\cdot i^4j + F^3(625i^6 + 32\cdot 48\cdot 8i^3j^2)\},\\
D &= 250\cdot 64F^6(i^3 - 6j^2).
\end{aligned} \tag{12}$$

These invariants are all functions of H, iF^2, and jF^3, except R which involves also T. Since

$$T^2 = -\tfrac{1}{2}(H^3 - \tfrac{1}{2}iHF^2 + \tfrac{1}{3}jF^3),$$

R^2 is a function of the same three quantities.

The quintic (11) having one root infinite and the sum of the other four zero is in precisely the form we considered earlier. Its roots can be identified with s_1, s_2, s_3, s_4, in (2) and expressed as functions of y_1, y_2, y_3. Equating the symmetric functions of s_i with the coefficients in (10) we find

$$\Sigma s_1 s_2 = -2\Sigma y_1^2 = 3H, \qquad \Sigma s_1 s_2 s_3 = -8 y_1 y_2 y_3 = -4T, \tag{13}$$
$$s_1 s_2 s_3 s_4 = \Sigma y_1^4 - 2\Sigma y_1^2 y_2^2 = \tfrac{1}{2} iF^2 - \tfrac{3}{4} H^2.$$

Since
$$2jF^3 = 3iF^2 H - 6H^3 - 12T^2,$$

$$H = -\tfrac{2}{3}\Sigma y_1^2, \qquad T = -2y_1 y_2 y_3, \qquad iF^2 = \tfrac{8}{3}(\Sigma y_1^4 - \Sigma y_1^2 y_2^2), \tag{14}$$
$$jF^3 = -\tfrac{8}{9}[2\Sigma y_1^6 - 3\Sigma y_1^4 (y_2^2 + y_3^2) + 12 y_1^2 y_2^2 y_3^2].$$

These values of iF^2, jF^3, H, T, in terms of y_1, y_2, y_3, substituted in (12) give the equations of the invariant curves A, B, C, D, R.

We can now show that any invariant curve of the group corresponds to an invariant of the quintic. Since the curve admits G_{24} it must be a function of Σy_1^2, $\Sigma y_2^2 y_3^2$, and $y_1 y_2 y_3$, or also of H, T, and iF^2. Hence at all points y on the curve the line yT is the root of a definite covariant of the four lines from y to the points i. But if the curve admits also a quadratic transformation like $T_{(04)}$, the same thing is true of the line $y4$ with regard to the four others, i. e., the covariant relation of the one line to the other four is an invariant relation of all five which is the same at all points of the curve.

In handling the invariant curves in later paragraphs we employ none of odd order and none of order as high as 54. Neither R nor R^2 can occur in the expressions used, and there is no loss of generality in making the calculation with $R = 0$ or with $y_1 = 0$. That is, the leading coefficient, or terms free of y_1, will be sufficient for our purposes. These are somewhat simpler if we take out of the invariant of degree $4k$ the factor 2^{5k}. A direct calculation gives the results:

$$A = 4(2, 1, 1, 2\,|\,y_2^2, y_3^2),$$
$$B = (1, 1, -30, 65, \bar{+}30, 1, 1\,|\,y_2^2, y_3^2),$$
$$B_1 = (-1, 2+5\delta_1, 2+5\delta_2, -1\,|\,y_2^2, y_3^2),$$
$$B_2 = (-1, 2+5\delta_2, 2+5\delta_1, -1\,|\,y_2^2, y_3^2),$$
$$C = -3(1, -11, 46, -86, 51, 51, -86, 46, -11, 1\,|\,y_2^2, y_3^2).$$

Here $B = B_1 B_2$. We have seen that B and C meet 10 times at each of the K-points, four of which are on the line $y_1 = 0$. The coördinates of these points are known and give the common factors of the leading coefficients of the curves B_i and C. In factored form the above coefficients are

$$A = 4(2y_2^4 - y_2^2y_3^2 + 2y_3^4)(y_2^2 + y_3^2),$$

$$B = (y_2^2 - \delta_1^2 y_3^2)^2(y_2^2 - \delta_2^2 y_3^2)^2(y_2^2 + \delta_1^4 y_3^2)(y_2^2 + \delta_2^4 y_3^2),$$

$$B_1 = -(y_2^2 - \delta_2^2 y_3^2)^2(y_2^2 + \delta_1^4 y_3^2), \tag{16}$$

$$B_2 = -(y_2^2 - \delta_1 y_3^2)^2(y_2^2 + \delta_2^4 y_3^2),$$

$$C = -3(y_2^2 - \delta_1^2 y_3^2)^4(y_2^2 - \delta_2^2 y_3^2)^4(y_2^2 + y_3^2).$$

Certain quintic resolvents of the general equation find an immediate geometric interpretation. The absolute invariants B/A^2 and C/A^3 are functions of $iH/jF = k_1$ and $i^3/j^2 = k_2$. Since k_1 and k_2 are invariant under G_{24}, they are five valued under G_{120} and the elimination of either leads to a quintic resolvent whose coefficients are invariants. Or if $\sqrt{D}$ be adjoined we can replace k_2 by $\frac{1}{9}\sqrt{\frac{5}{2}}(k_2 - 6) = \nu$. To obtain a simple result we shall introduce new absolute invariants J and L. Let

$$\frac{B_1}{A} = \frac{5L - 1}{8}, \qquad \text{whence} \qquad k_2 = \frac{3\nu + L}{4L},$$

and

$$75J = \left(-4 \cdot 24^3 \frac{C}{A^3} - 24 + 27 \cdot 25\, L^2\right);$$

then the elimination of k_1 and substitution of ν for k_2 gives the resolvent

$$J\nu^5 + L(4 - 125L^2)\nu^4 + \tfrac{5}{2}\tfrac{5}{7}(J + 153L^2 - 4)\nu^3 - 50L^3\nu^2 + 5L^2\nu - 5L^3 = 0. \tag{17}$$

Since $j = 0$ if $\nu = \infty$, the numerator of the absolute invariant J of degree twelve is the condition that four of the roots be harmonic. L is proportional to ν and the $\sqrt{D}$ formed for the four roots of the quintic. If (17) is made integral in the invariants by multiplication with A^3, the resulting curve, for a given value of ν, is of order 18. It is the product of the twelve lines and three conics determined by requiring that four of the roots of a quintic or any even permutation of them have a given anharmonic ratio.

§ 3. *The form problems associated with the Cremona group.*

Under the group of § 1 there is a net of invariant curves of order thirty, $\rho A^3 + \sigma AB + \tau C = 0$. If numerical values are assigned to A, B, C, then ρ, σ, τ, are subject to a linear condition and a pencil of the net is determined. The various pencils in the net have 900 base points, but 780 of these are the same for all pencils. They are the $5 \cdot 12 \cdot 12 = 720$ meets at the points p_i and the $10 \cdot 30 - 5 \cdot 4 \cdot 12$ meets of A and C outside the points p_i. Hence only 120 of the base points of pencils formed from the net are movable. They are, of course, a set of 120 conjugate points. Instead of numerical values of A, B, C, it is sufficient to give the values of $B/A^2 = k$ and $C/A^3 = \lambda$. The pencil is

then fixed by the two curves $A(B-kA^2)=0$ and $C-\lambda A^3=0$. The form problem of the group can be stated as follows:

Given the numerical values of A, B, C, *or of* B/A^2, C/A^3, *to find the ratios* $x_1:x_2:x_3$ *for the* 120 *points* x *for which the functions* A, B, C, *or the ratios* B/A^2, C/A^3, *take the assigned values.*

One point x is sufficient, since the others are determined from it by a set of substitutions known in advance. The problem will be referred to as the *I*-form-problem. Its solution is equivalent to the solution of a given quintic to within rational operations. For a given quintic fixes the values of A, B, C. The solution of the form problem fixes a point x such that the five lines $\overline{xp_i}$ are projective to the given quintic. The known quintic can be transformed into the given one by a known linear substitution determined by their linear covariants. The problem has exactly the same statement for the group in § 2.

If $\sqrt{D}$ be adjoined as a known quantity, i. e., if $\Pi(\overline{p_ip_k})$ is given a definite sign, the group reduces to the alternating G_{60} and the invariant curves are expressible by means of B_1, B_2, C, and R. In this case we consider the net of curves of order 30, $B_1^2(\rho B_1+\sigma B_2)+\tau C=0$. The pencils in the net have 840 fixed base points, 720 at the points p_i and 120 meets of B_1^2 and C outside the points p_i. The remaining 60 base points are movable. The form problem now reads:

Given the numerical values of B_1, B_2, C *or of* B_2/B_1, C/B_1^3, *to find the ratios* $x_1:x_2:x_3$ *for the* 60 *points* x *for which* B_1, B_2, C, *or* B_2/B_1, C/B_1^3, *take the assigned values.*

In place of B_1 and B_2, A and $\sqrt{D}$ can be given. They are connected by the equations $4(B_1+B_2)=-A$ and $4(B_1-B_2)=\sqrt{D}$. As before the solution of this form problem — referred to as the I'-form problem — carries with it the solution of a binary quintic for which $\sqrt{D}$ is given.

§ 4. *The reduction of the I'-form problem to Klein's "problem of the A's."*

The expressions in the *Ikosaeder* denoted by A_0, A_1, A_2; A_0', A_1', A_2'; A, B, C, D, will be referred to by the corresponding small letters to avoid confusion with the notation already introduced. In this paragraph the group of § 2 is employed.

Through five points in space $q_0, q_1, \cdots, q_4$ there pass ∞^2 cubic curves. On each curve the five points determine a system of projective quintics and a given quintic whose roots are ordered with regard to the five points determines a definite curve. There is thus a one-to-one correspondence between the cubic curves and the systems of projective quintics or also a one-to-one correspondence between the cubic curves and the point y_1, y_2, y_3 of the plane (§ 2). To obtain the cubic curve determined by the point y, we take the five points in space with the coördinates suggested in the *Ikosaeder*, p. 187; the coördinates of q_i are

$$p_1 : p_2 : p_3 : p_4 = \epsilon^i : \epsilon^{2i} : \epsilon^{3i} : \epsilon^{4i}. \tag{1}$$

Let the parameters of the five points in order on the cubic curve be $t = \infty$, s_1, s_2, s_3, s_4, and let $c_i = (t - s_k)(t - s_l)(t - s_m)$, $i, k, l, m = 1, 2, 3, 4$. The parametral equation of the cubic is then

$$p_i = \epsilon^i c_1 + \epsilon^{2i} c_2 + \epsilon^{3i} c_3 + \epsilon^{4i} c_4. \tag{2}$$

The c_i as functions of s_i and therefore of y_i give the connection sought.

If the point y be subjected to the Cremona G_{120}, the corresponding cubic curve takes 120 positions. The 120 curves are conjugate under the collineation group, g_{120}, of the points q_i. Linear complexes are also linearly transformed by g_{120} and there arises in this way an isomorphic collineation group on the six coördinates A_{ik}. KLEIN shows (*Ikosaeder*, pp. 176–181 and pp. 245–247) that if the following combinations

$$2a_0 = A_{23} + A_{41}, \qquad a_1 = A_{42}, \qquad a_2 = A_{31},$$
$$2a_0' = -A_{23} + A_{41}, \qquad a_1' = -A_{43}, \qquad a_2' = A_{12},$$

of the complex coördinates be employed, the new coördinates are transformed by the even subgroup g_{60} of g_{120} in such a way that a_0, a_1, a_2, alone experience the linear transformations of the ternary ikosahedral group and a_0', a_1', a_2', alone experience the transformations of the "contragredient" form of the same group (that form in which ϵ is replaced by ϵ^2).

The cubic curve (2) defines a linear complex and we shall find, first of all, the coördinates a_0, a_1, a_2, a_0', $a_1'a_2'$ of this complex. Two planes, $\sum_{i=1}^4 \rho_i p_i$ and $\sum_{l=1}^4 \sigma_l p_l$, cut the cubics in 3 points each, whose parameters are given by the binary cubics in t:

$$\sum_{i,\,k=1}^{4} \epsilon^{ik} \rho_i c_k = 0 \qquad \text{and} \qquad \sum_{l,\,m=1}^{4} \epsilon^{lm} \sigma_l c_m = 0.$$

These cubics are apolar, and the line of the planes is a line of the complex, if

$$\sum_{k,\,m} \{ \sum_i \epsilon^{ik} \rho_i \sum_l \epsilon^{lm} \sigma_l - \sum_i \epsilon^{im} \rho_i \sum_l \epsilon^{lk} \sigma_l \} (c_k, c_m) = 0$$

where $(c_k, c_m) = -(c_m, c_k) = (s_k - s_m)(s_i - s_j)^2$ is the apolarity condition of the cubics c_k and c_m. The summation can be put in the form

$$\sum_{k,\,m} (c_k, c_m) \{ \sum_{i,\,l} (\rho_i \sigma_l - \rho_l \sigma_i)(\epsilon^{ik+lm} - \epsilon^{im+lk}) \} = 0,$$

or expressed in planar linear coördinates Π_{il}, the equation of the complex is

$$\sum_{i,\,l} \Pi_{il} [\sum_{k,\,m} (\epsilon^{ik+lm} - \epsilon^{im+lk})(c_k, c_m)] = 0. \tag{3}$$

The coefficient of Π_{il} is A_{il}, and (c_k, c_m) is the product of the line joining the points k and m and the square of the line joining the other two points. The

six cubics (c_k, c_m) are therefore linearly independent plane cubics through the points 1, 2, 3, 4. It is convenient to replace these six by the following two sets of three:

$$
\begin{aligned}
C_1 &= y_1(y_1^2 + \delta_2 y_2^2 + \delta_1 y_3^2), & K_1 &= y_1(y_1^2 + \delta_1 y_2^2 + \delta_2 y_3^2),\\
C_2 &= y_2(\delta_1 y_1^2 + y_2^2 + \delta_2 y_3^2), & K_2 &= y_2(\delta_2 y_1^2 + y_2^2 + \delta_1 y_3^2),\\
C_3 &= y_3(\delta_2 y_1^2 + \delta_1 y_2^2 + y_3^2), & K_3 &= y_3(\delta_1 y_1^2 + \delta_2 y_2^2 + y_3^2),
\end{aligned} \tag{4}
$$

In terms of these we have

$$
\begin{aligned}
a_0 &= (\epsilon^2 - \epsilon^3)C_2 + (\epsilon^4 - \epsilon)C_3,\\
a_1 &= -(\epsilon^4 - \epsilon)(\epsilon^2 - \epsilon^3)C_1 + (\epsilon^4 - \epsilon)C_2 - (\epsilon^2 - \epsilon^3)C_3,\\
a_2 &= (\epsilon^4 - \epsilon)(\epsilon^2 - \epsilon^3)C_1 + (\epsilon^4 - \epsilon)C_2 - (\epsilon^2 - \epsilon^3)C_3,\\
a_0' &= (\epsilon^4 - \epsilon)K_2 - (\epsilon^2 - \epsilon^3)K_3,\\
a_1' &= (\epsilon^4 - \epsilon)(\epsilon^2 - \epsilon^3)K_1 - (\epsilon^2 - \epsilon^3)K_2 - (\epsilon^4 - \epsilon)K_3,\\
a_2' &= -(\epsilon^4 - \epsilon)(\epsilon^2 - \epsilon^3)K_1 - (\epsilon^2 - \epsilon^3)K_2 - (\epsilon^4 - \epsilon)K_3.
\end{aligned} \tag{5}
$$

Formulas (4) and (5) give the coördinates of the complex of the cubic space curve determined by the point y. Also *for every complex arising from a cubic space curve* we find one point y, since a_i and a_i' determine C_i and K_i, and from the equations

$$\frac{C_1}{y_1} + \frac{C_2}{y_2} + \frac{C_3}{y_3} = 0, \qquad \frac{K_1}{y_1} + \frac{K_2}{y_2} + \frac{K_3}{y_3} = 0,$$

we have

$$y_1 : y_2 : y_3 = \frac{1}{C_2K_3 - C_3K_2} : \frac{1}{C_3K_1 - C_1K_3} : \frac{1}{C_1K_2 - C_2K_1}. \tag{6}$$

As the variables y are subjected to the operations * of the Cremona G_{60}, the a_i and a_i' are subjected to the collineations of the two forms of the ternary ikosahedral group. Hence

(7) *Any invariant of the group of the a_i, or any simultaneous invariant of the group of the a_i and the group of the a_i', when expressed in terms of y is an invariant of the Cremona G_{60}, and therefore a rational function of B_1, B_2, C.*

It is necessary, for our purpose, to find in terms of B_1, B_2, C, the invariants a, b, c (*Ikosaeder*, pp. 214–8) of orders 2, 6, 10 in a_i and the simultaneous invariants F_1, F_2, F_3 (*Ikosaeder*, pp. 231–2) linear in a_i' and of orders 3, 5, 7 respectively in a_i. The respective orders in y are 6, 18, 30, and 12, 18, 24.

* The cubics C_i and K_i are transformed into cubics multiplied by a factor which is the product of the three lines of the principal triangle. But when only the ratios are under consideration, this factor, which is the same for all six cubics, can be dropped.

Since these are all even and of order less than 54, it will be sufficient to use the leading coefficient in identifying them.

When $y_1 = 0$,

$$a_0 = (\epsilon^2 - \epsilon^3)(1, -1, \delta_2, \delta_2 \mid y_2, y_3),$$

$$(8) \quad a_1 = a_2 = (\epsilon^2 - \epsilon^3)(\delta_2, -\delta_1, \delta_2^2, -1 \mid y_2, y_3) = (\epsilon^4 - \epsilon)(y_2 - \delta_2 y_3)(y_2 - \delta_1 y_3)^2,$$

$$a_0' = (\epsilon^4 - \epsilon)(1, -1, \delta_1, \delta_1 \mid y_2, y_3),$$

$$(9) \quad a_1' = a_2' = (\epsilon^4 - \epsilon)(\delta_1, -\delta_2, \delta_1^2, -1 \mid y_2, y_3) = -(\epsilon^2 - \epsilon^3)(y_2 - \delta_1 y_3)(y_2 - \delta_2 y_3)^2,$$

$$a = a_0^2 + a_1^2,$$

$$(10) \quad b = (8a_0^4 - 2a_0^2 a_1^2 - 2a_0 a_1^3 + a_1^4) a_1^2,$$

$$c = 4(80a_0^6 - 63a_0^5 a_1 - 40a_0^4 a_1^2 + 40a_0^3 a_1^3 + 5a_0^2 a_1^4 - 10a_0 a_1^5 + 2a_1^6) a_1^4,$$

$$F_1 = 2a_0' a_0 (2a_0^2 - 3a_1^2) - 2a_1' a_1^2 (3a_0 + a_1),$$

$$F_2 = 4a_0' (-4a_0^3 + 3a_0 a_1^2 - a_1^3) a_1^2 + 4a_1' (8a_0^3 - 4a_0^2 a_1 - 2a_0 a_1^2 + a_1^3) a_1^2,$$

$$F_3 = 4a_0' (16a_0^3 - 4a_0^2 a_1 - 8a_0 a_1^2 + 3a_1^3) a_1^4 + 4a_1' (-16a_0^5 + 24a_0^4 a_1 - 16a_0^3 a_1^2 - 2a_0^2 a_1^3 + 7a_0 a_1^4 - 2a_1^5) a_1^2.$$

The values of a and F_1 are not hard to find by direct calculation. The following method reduces considerably the difficulty of finding the values of b, c, F_2, F_3. Call the number of times that a term contains the factor $(y_2 - \delta_1 y_3)$ its "first rank"; the number of times it contains the factor $(y_2 - \delta_2 y_3)$ its "second rank"; and let $\{i, k\}$ be the "rank" of the term if its first and second ranks are i and k respectively. Then a_0, a_1, a_0', a_1'; C, B_2, B_1, have the ranks $\{0, 0\}$, $\{2, 1\}$, $\{0, 0\}$, $\{1, 2\}$; $\{4, 4\}$, $\{2, 0\}$, $\{0, 2\}$ respectively. Let

$$F_3 = \mu_1 CB_1 + \mu_2 CB_2 + \mu_3 B_1^2 B_2^2 + \mu_4 B_1^3 B_2 + \mu_5 B_1 B_2^3 + \mu_6 B_1^4 + \mu_7 B_2^4.$$

Now B_2^4 and $B_1^3 B_2$ are the only terms (including of course the terms in the expression for F_3 in (11)) which have the second rank 0 and 2 respectively, hence $\mu_5 = \mu_7 = 0$. For a similar reason with regard to the first rank, $\mu_4 = \mu_6 = 0$. μ_1/μ_3 is determined by making the first rank of $C + kB_1 B_2^2$ five. We find that $k = -\frac{3}{4}$. Then μ_1 and μ_3 are obtained by making the first rank of $16a_1' a_0^5 a_1^2 + \mu_1 (CB_1 + kB_1^2 B_2^2)$ equal to 6. The result is $\mu_1 = \frac{3}{8} 5^3$, $\mu_3 = -8 \cdot 5^3$. Finally μ_2 is obtained by requiring the term

$$-16a_0' a_0^3 a_1^4 + 16a_1' a_0^5 a_1^2 + \mu_2 CB_2 - 8 \cdot 5^3 B_1^2 B_2^2$$

to have a second rank 5, which determines μ_2 to be $4 \cdot 5^3$. We have determined thus

$$F_3 = 2^2 \cdot 5^3[\tfrac{8}{3} CB_1 + CB_2 - 2B_1^2 B_2^2].$$

The quantity c is found much more easily though of higher degree. Let

$$c = \nu_1 CB_1^2 + \nu_2 CB_1 B_2 + \nu_3 CB_2^2 + \nu_4 B_2^3 B_1^2 + \nu_5 B_2^2 B_1^3 + \nu_6 B_2^4 B_1 + \nu_7 B_2 B_1^4 + \nu_8 B_1^5 + \nu_9 B_2^5.$$

The lowest rank occurring in the expression for c in (10) is $\{8, 4\}$. Noting that there are only single terms of first rank 0 and 2 and of second rank 0 and 2 we have $\nu_6 = \nu_7 = \nu_8 = \nu_9 = 0$. ν_1 and ν_5 must be chosen so that the first rank of $B_1^2(\nu_1 C + \nu_5 B_1 B_2^2)$ is 6. But this is impossible for we have just seen that the first rank of $\nu_1 C + \nu_5 B_1 B_2^2$ can be reduced to five only by taking $\nu_5/\nu_1 = -\frac{3}{4}$. Hence $\nu_5 = \nu_1 = 0$. Then ν_2 and ν_4 must be chosen so that the first rank of $B_1 B_2(\nu_2 C + \nu_4 B_1 B_2^2)$ shall reduce from 6 to 8; this again is impossible. Hence $\nu_2 = \nu_4 = 0$. Finally we determine ν_3 so that the rank (8, 4) of $-4 \cdot 80 a_0^6 a_1^4 + \nu_3 CB_2^2$ shall increase to $\{10, 5\}$. We find $\nu_3 = 5^5 \cdot 2^2/3$ and $c = 5^5 \cdot 2^2/3\, CB_2^2$.

Proceeding similarly with the simpler forms b and F_2 we obtain all the desired expressions as below:

$$a = 4B_1 + B_2,$$

(12)
$$b = 25(\tfrac{4}{3} C + B_1 B_2^2),$$

$$c = 2^2 \cdot 5^5 \cdot \tfrac{1}{3} CB_2^2;$$

$$F_1 = 2(16B_1^2 + 4B_2^2 + 13B_1 B_2),$$

(13)
$$F_2 = 2^2 \cdot 5(-6C + 5B_1 B_2^2),$$

$$F_3 = 2^2 \cdot 5^3(\tfrac{8}{3} CB_1 + CB_2 - 2B_1^2 B_2^2).$$

To solve the I'-form problem we find, from the given quantities B_1, B_2, C, the values of a, b, c, and F_1, F_2, F_3 in (12) and (13). Taking the "problem of the a's" in this form:

(14) *Given the numerical values of a, b, c or of the ratios b/a^3, c/a^5, to find the ratios of the coördinates of the* 60 *points $a_0 : a_1 : a_2$ for which a, b, c or b/a_3, c/a^5 take the assigned values;*

we have from the values of a, b, c, the values of $a_0 : a_1 : a_2$. As F_1, F_2, F_3, are linear in a_0', a_1', a_2', their numerical values and those of a_0, a_1, a_2 determine a_0', a_1', a_2'. The values of a_i and a_i' fix the values of C_i and K_i in (5). These in turn, in (6), determine the required solution, $y_1 : y_2 : y_3$.

There is, however, one case in which the method just given does not apply. The three linear equations used to obtain a_i' are of degrees 3, 5, 7 in a_i. The determinant of the system is an invariant of degree 15 and must be the curve

d (*Ikosaeder*, p. 218). The curve d is the product of the fifteen axes of the harmonic perspectivities in the group of the a_i, one of which is $a_1 - a_2 = 0$. But if $a_1 - a_2 = 0$, $C_1 = 0 = y_1(y_1^2 + \delta_2 y_2^2 + \delta_1 y_3^2)$. If $y_1 = 0$, $R = 0$. But if y is a point on the conic $y_1^2 + \delta_2 y_2^2 + \delta_1 y_3^2 = 0$, a certain alternating invariant of degree 12 vanishes. The corresponding curve is the product of the three conics and twelve lines into which the given conic is transformed by G_{60}. Calling this curve S_1 and identifying it by means of its leading coefficient as before we find that

$$S_1 = 32C - 24\ B_1 B_2^2 + 3B_2^3. \tag{15}$$

Its transform by an odd substitution of G_{120} is

$$S_2 = 32C - 24B_1^2 B_2 + 3B_1^3. \tag{16}$$

If $R = 0$, due to $y_1 = 0$, both $a_1 = a_2$ and $a_1' = a_2'$, while if $S_1 = 0$, due to $y_1^2 + \delta_2 y_2^2 + \delta_1 y_3^2 = 0$, then $a_1 = a_2$; but $a_1' \neq a_2'$ except for a finite number of points. We suppose (see § 6) that $R \neq 0$. If $S_1 = 0$, the given quantities B_1 and B_2 can be interchanged, and, provided also $S_2 \neq 0$, the solution can be carried through as before. The resulting $y_1 : y_2 : y_3$ must then be transformed by an odd substitution of G_{120} to afford a solution for the constants originally given. If $S_1 = 0$ and $S_2 = 0$, the form problem can be solved directly by finding the intersections of these two curves whose factors are linear or quadratic.

The invariant

$$S = S_1 S_2 = \tfrac{1}{2}\left[2^{11}C^2 + 2^4 \cdot 33\,CBA - 3\,CA^3 + 2 \cdot 3^6 B^3 - 3^2 B^2 A^2\right]$$

has for corresponding curve the locus of points whose quintics are of the type $(z_1 + az_2)(z_1^5 - z_2^5)/(z_1 - z_2)$, i. e., four of the roots belong to a cyclo-projective set of five.

§ 5. *A new solution of the "Problem of the A's."*

We have considered the I'-form problem and the a-form problem as depending essentially on two parameters and have avoided the use of the invariant curves R and d. The solution of the a-problem suggested by KLEIN (*Ikosaeder*, pp. 234–8) involves, at least explicitly, the use of d. The solution here given is, in some respects, quite similar, but the point of view is more nearly that of the third chapter of the second part of the *Ikosaeder* where the "principal equation" is solved by the use of a singly infinite *linear* system of curves on the principal surface (pp. 190–4). The a-problem can be solved by the extraction of a square root and a root of the ikosahedral equation. We might infer from this *the existence of singly infinite quadratic systems of invariant rational curves* and, in fact, the simplest system of that sort is found below and employed to effect the solution. All the necessary rational formulae will be given explicitly.

It is desirable, first of all, to obtain expressions that will include all rational invariant curves. If such a curve is of order n we can take it to be

$$U: \quad [a_0 = g_0^{(n)}(\nu), \qquad a_1 = g_1^{(n)}(\nu), \qquad a_2 = g_2^{(n)}(\nu)], \tag{1}$$

$g_i^{(n)}(\nu)$ being a binary form of degree n in the variables ν_1, ν_2. If U is invariant under g_{60}, the group of the a's, to every member of g_{60} must correspond a transformation of the binary variables, and the totality of such transformations will be the binary ikosahedral group, γ_{60}. This group being put in the canonical form, there are, as KLEIN points out (*Ikosaeder*, p. 232), two possible cases; γ_{60} is either "cogredient" or "contragredient" to g_{60}. In the first case the parameter is called λ_1, λ_2, in the second μ_1, μ_2. The point $a_0 : a_1 : a_2$ is given by the coefficients of the quadratic form $a_1\lambda_1'^2 + 2a_0\lambda_1'\lambda_2' - a_2\lambda_2'^2$, an invariant if the a_i are transformed by g_{60} and λ_1', λ_2' are simultaneously transformed by the corresponding substitutions of the cogredient form of γ_{60}. Putting in this quadratic the values of a_i from (1) we have

$$g_1^{(n)}(\nu)\lambda_1'^2 + 2g_0^{(n)}(\nu)\lambda_1'\lambda_2' - g_2^{(n)}(\nu)\lambda_2'^2 \tag{2}$$

and *this must be an invariant under γ_{60} in the cogredient variables λ and λ' of degrees n and 2 respectively* or *in the contragredient variables μ and λ' of degrees n and 2 respectively. Conversely any such form determines a rational invariant curve.*

In the first case the form (2) is a rational and integral function of the ikosahedral invariants, $f(\lambda)$, $H(\lambda)$, $T(\lambda)$; their first and second polars as to λ', and $\lambda_1\lambda_2' - \lambda_2\lambda_1'$, the function being homogeneous and of the proper degree in both variables. In the second case, the form (2) is a rational and integral function of $f(\mu)$, $H(\mu)$, $T(\mu)$, four invariants * linear in λ' and of degrees 7, 13, 17, 23 in μ, and three invariants, quadratic in λ' and of degrees 6, 10, 16 in μ, the function being homogeneous and of the proper degrees. If f' and H' are the first polars of f and H as to λ', the form (2) which defines the system used hereafter is

$$(\lambda_1\lambda_2' - \lambda_2\lambda_1')(9\rho f H' - 25 H f'), \tag{3}$$

ρ being the parameter of the system.

The system itself is suggested by the following considerations. The coördinates of a point, as the system of coefficients of a quadratic, determine values of λ and λ', the parameters of the two points of contact of tangents from the point to the conic, $a = a_0^2 + a_1 a_2$, i. e.,

$$2a_0 = -(\lambda_1\lambda_2' + \lambda_2\lambda_1'), \qquad a_1 = \lambda_2\lambda_2', \qquad a_2 = -\lambda_1\lambda_1'. \tag{4}$$

*See the complete system, established by GORDAN, Mathematische Annalen, vol. 13 (1878), p. 386, for the invariants containing contragredient variables λ' and μ.

If λ and λ' are simultaneously transformed by γ_{60}, $a_0 : a_1 : a_2$ is transformed by g_{60}. The point a_i depends on two quantities, *both variable* under γ_{60}. If one of the quantities can be expressed in terms of the other and a third quantity ρ which is *invariant* under γ_{60}, we should have in (4) the equation of a system of invariant rational curves. This is possible, for example, by means of the equations

$$\begin{aligned} 12f' &= \frac{\partial f}{\partial \lambda_1}\lambda_1' + \frac{\partial f}{\partial \lambda_2}\lambda_2' = 3f\rho_1 = r, \\ 20H' &= \frac{\partial H}{\partial \lambda_1}\lambda_1' + \frac{\partial H}{\partial \lambda_2}\lambda_2' = 5H\rho_2 = s. \end{aligned} \tag{5}$$

So defined ρ_1 and ρ_2 are invariant under γ_{60}. Solve equations (5) for λ_1' and λ_2' and substitute their values in (4), and we have the desired system of rational curves U:

$$\begin{aligned} 2a_0 &= \rho_1 3f\left(\lambda_2\frac{\partial H}{\partial \lambda_2} - \lambda_1\frac{\partial H}{\partial \lambda_1}\right) - \rho_2 5H\left(\lambda_2\frac{\partial f}{\partial \lambda_2} - \lambda_1\frac{\partial f}{\partial \lambda_1}\right), \\ a_1 &= \rho_1 3f\left(\lambda_2\frac{\partial H}{\partial \lambda_1}\right) \qquad - \rho_2 5H\left(\lambda_2\frac{\partial f}{\partial \lambda_1}\right), \\ a_2 &= \rho_1 3f\left(\lambda_1\frac{\partial H}{\partial \lambda_2}\right) \qquad - \rho_2 5H\left(\lambda_1\frac{\partial f}{\partial \lambda_2}\right); \end{aligned} \tag{6}$$

or again

$$\begin{aligned} a_0 &= \rho_1 f\cdot 6(\lambda_1^{20} - 114\lambda_1^{15}\lambda_2^5 - 114\lambda_1^5\lambda_2^{15} - \lambda_2^{20}) - \rho_2 H\cdot 5\lambda_1\lambda_2(-\lambda_1^{10} - \lambda_2^{10}), \\ a_1 &= \rho_1 f\cdot 12\lambda_1^4\lambda_2(-\lambda_1^{15} + 171\lambda_1^{10}\lambda_2^5 - 247\lambda_1^5\lambda_2^{10} - 57\lambda_2^{15}) \\ &\qquad - \rho_2 H\cdot\lambda_2^2(11\lambda_1^{10} + 66\lambda_1^5\lambda_2^5 - \lambda_2^{10}), \\ a_2 &= \rho_1 f\cdot 12\lambda_1\lambda_2^4(57\lambda_1^{15} - 247\lambda_1^{10}\lambda_2^5 - 171\lambda_1^5\lambda_2^{10} - \lambda_2^{15}) \\ &\qquad - \rho_2 H\cdot\lambda_1^2(\lambda_1^{10} + 66\lambda_1^5\lambda_2^5 - 11\lambda_2^{10}). \end{aligned} \tag{7}$$

That this is a quadratic system of curves follows at once from the definition of the parameter λ as one of the *two* parameters λ and λ' in (4). The order of the curves is, in general, 32. But when $\rho_1 = 0$, the order is 12; when $\rho_2 = 0$, the order is 20. Also when $\rho_1 = \rho_2$, the order is 2. For then $a_0 = \lambda_1\lambda_2 T$, $a_1 = -\lambda_2^2 T$, $a_2 = \lambda_1^2 T$. The curve is $a = a_0^2 + a_1 a_2$. This is seen at once by forming $4(a_0^2 + a_1 a_2)$ in (6) and making use of Euler's theorem. We find

$$4(a_0^2 + a_1 a_2) = 3600 f^2 H^2(\rho_1 - \rho_2)^2. \tag{8}$$

The equation of the general curve $\mathbf{U}$ is quadratic in ρ_1, ρ_2. For a fixed point $a_0 : a_1 : a_2$, the two values of ρ_1/ρ_2 are obtained from (5), as it stands, and from interchanging λ and λ' in (5). Denote by $\bar{g}$ the value that g takes when λ and λ' are interchanged. Then

$$\rho_1\bar{\rho}_1 = \frac{r\bar{r}}{9f\bar{f}}, \qquad \rho_2\bar{\rho}_2 = \frac{s\bar{s}}{25H\bar{H}}, \qquad \rho_1\bar{\rho}_2 + \rho_2\bar{\rho}_1 = \frac{Hr\bar{f}\bar{s} + \bar{H}\bar{r}fs}{15f\bar{f}H\bar{H}},$$

and ρ_1, ρ_2 satisfy the equation

$$9f\bar{f}\cdot s\bar{s}\rho_1^2 - 15(Hr\bar{f}\bar{s} + \bar{H}\bar{r}fs)\rho_1\rho_2 + 25H\bar{H}\cdot r\bar{r}\rho_2^2 = 0.$$

The coefficients being symmetric in λ and λ' of degree 32 in each are functions of a_0, a_1, a_2 of degree 32. Being invariant as well they are functions of a, b, c. The equation can be made more symmetrical by obtaining a new middle coefficient. When $\rho_1 = \rho_2$ the equation must reduce to $k\cdot T\bar{T}\cdot a = 0$. By equating the coefficients of $\lambda_1^{32}\lambda_2'^{32}$, k is found to be -25.

The new equation reads

$$(9)\quad 9f\bar{f}\cdot s\bar{s}\cdot\rho_1^2 - (9f\bar{f}\cdot s\bar{s} + 25H\bar{H}\cdot r\bar{r} + 25T\bar{T}\cdot a)\rho_1\rho_2 + 25H\bar{H}\cdot r\bar{r}\rho_2^2 = 0.$$

We have therefore to calculate $f\bar{f}$, $H\bar{H}$, $T\bar{T}$, $r\bar{r}$, $s\bar{s}$, and solve the equation (9) to obtain a value of $\rho = \rho_1/\rho_2$. We have further to find $Z_{(\rho)}$, the ikosahedral parameter on the curve $U_{(\rho)}$ fixed by ρ. Denote by $a_{(\rho)}$, $b_{(\rho)}$, $c_{(\rho)}$, the values of a, b, c, on the curve $U_{(\rho)}$. The curves $r\bar{r} = 0$ and $s\bar{s} = 0$ are $U_{(0)}$ and $U_{(\infty)}$ respectively. For $U_{(0)}$ we find

$$(10)\quad \begin{aligned} a_{(0)} &= 36f^2, \\ b_{(0)} &= f(12^3\cdot 32f^5 - 5H^3), \\ c_{(0)} &= (16\cdot 12^3f^5)^2 - 11(16\cdot 12^3f^5)H^3 + H^6, \end{aligned}$$

Therefore

$$\frac{b_{(0)}}{a_{(0)}^3} = \frac{1}{27}(32 - 5Z_{(0)}), \qquad \frac{3^4}{4}\frac{c_{(0)}}{a_{(0)}^5} = 2^8 - 11\cdot 2^4Z_{(0)} + Z_{(0)}^2.$$

Solving for $Z_{(0)}$, then eliminating $Z_{(0)}$ to obtain the equation of the curve we have

$$(11)\quad \begin{aligned} Z_{(0)} &= \frac{32a^3 - 27b}{5a^3}, \\ 16r\bar{r} &= 4(9b^2 + 16\cdot 17a^3b - 16^2a^6) - 25ac. \end{aligned}$$

The coefficient of $r\bar{r}$, 16, is determined from the highest term. Similarly for $U_{(\infty)}$, we find

$$(12)\quad \begin{aligned} a_{(\infty)} &= 3\cdot 12H^2, \\ b_{(\infty)} &= 12^6f^5(28H^3 - 12^3f^5), \\ c_{(\infty)} &= \tfrac{1}{4}12^{10}\cdot f^5H(-16H^6 + 40H^3\cdot 12^3f^5 + 3\cdot 12^6f^{10}), \end{aligned}$$

Therefore

$$\frac{b_{(\infty)}}{a^3_{(\infty)}} = \frac{1}{27 Z_{(\infty)}}\left(28 - \frac{1}{Z_{(\infty)}}\right), \qquad \frac{c_{(\infty)}}{a^5_{(\infty)}} = \frac{4}{27 Z_{(\infty)}}\left(-16 + \frac{40}{Z_{(\infty)}} + \frac{3}{Z^2_{\infty}}\right);$$

$$Z_{(\infty)} = \frac{4a^2(128a^3 - 3b)}{c + 496a^2 b}, \tag{13}$$

$$s\bar{s} = 2^7 \cdot 5^2(c^2 + 2^4 \cdot 8^3 ca^2 b + 2^4 \cdot 3^5 ab^3 - 2^{11} \cdot 7ca^5 + 2^{10} \cdot 79a^4 b^2 - 2^{15} \cdot a^7 b).$$

The value of $Z_{(1)}$ is given by KLEIN (*Ikosaeder*, p. 221),

$$Z_{(1)} = \frac{c^3}{12^3 b^5}. \tag{14}$$

To every value of ρ corresponds a definite U_ρ and on it a definite $Z_{(\rho)}$. Hence $Z_{(\rho)}$ is a rational function of ρ. Since ρ satisfies the quadratic equation (9), there is no loss in assuming $Z_{(\rho)}$ to be a linear function of ρ. And the values of $Z_{(\rho)}$ being known for three values of ρ we have in general:

$$Z_{(\rho)} = \frac{Z_\infty(Z_{(0)} - Z_{(1)})\rho + Z_{(0)}(Z_{(1)} - Z_{(\infty)})}{(Z_{(0)} - Z_{(1)})\rho + (Z_{(1)} - Z_{(\infty)})}. \tag{15}$$

There remains only the calculation of $f\bar{f}$, $H\bar{H}$, $T\bar{T}$, in terms of a, b, c. They are respectively the products of the 12, 20, 30 tangents at the f, H, and T points on a. By direct calculation we find

$$f\bar{f} = (115b + 128a^3)b,$$

$$H\bar{H} = c^2 + 2^{20}a^{10} - 2^{17} \cdot 5a^7 b + 2^9 \cdot 5 \cdot 241a^4 b^2 - 2^9 \cdot 101a^5 c - 2^5 \cdot 131cba^2 - 2^5 \cdot 3359ab^3. \tag{16}$$

The quickest way to get $T\bar{T}$ is to suppose $2a_0 = i(a_1 + a_2)$, as this is one of the lines of the curve. Writing $a_1 + a_2 = 2\alpha$, $a_1 - a_2 = 2\beta$; a, b, c reduce to functions of α^2 and β^2. Finally putting $(1 - 2i)\alpha^2 = k$, $\beta^2 = l$; we find for a, b, and c the values

$$a = -l, \qquad b = -k^3 + 5k^2 l + 5kl^2 - l^3,$$

$$c = 4(-3k^5 - 45k^4 l + 10k^3 l^2 - 50k^2 l^3 + 25kl^4 - k^5).$$

The identity of degree 15 connecting these forms is to be found. This gives

$$\begin{aligned} T\bar{T} = {} & c^3 - 12^3 b^5 - \tfrac{1}{2}3 \cdot 11 \cdot 17c^2 a'^5 - 2 \cdot 3 \cdot 5 \cdot 11c^2 ba'^2 + 2 \cdot 3 \cdot 23ca'^{10} \\ & - 2 \cdot 3 \cdot 5^2 \cdot 17cba'^7 + 2^4 \cdot 3 \cdot 5 \cdot 11cb^2 a'^4 - 2^3 \cdot 3^2 \cdot 5 \cdot 7cb^3 a' \\ & - 2^3 \cdot 3^2 \cdot 5 \cdot 349b^4 a'^3 + 2^3 \cdot 5 \cdot 691b^3 a'^6 - 2^2 \cdot 3 \cdot 5 \cdot 37b^2 a'^9 \\ & - 2^2 \cdot 3 \cdot 5ba'^{12} + a'^{15} \end{aligned} \tag{17}$$

where $a' = 4a$.

Briefly, then, to solve the problem of the a's we find, from formulas (11), (13), (14), (16), (17), the values of $f\bar{f}$, $H\bar{H}$, $T\bar{T}$, $r\bar{r}$, $s\bar{s}$, $Z_{(0)}$, $Z_{(\infty)}$, $Z_{(1)}$ from the given values of a, b, c. By the extraction of a square root, a value of ρ is found in (9). The value of $Z_{(\rho)}$ is given by (15), and from it the ikosahedral irrationality λ_1/λ_2 is calculated. Finally the values of $a_0 : a_1 : a_2$ are obtained from (7).*

§6. *The solution of the quintic equation.*

We have now all the material at hand to outline a series of operations that will effect the solution of the general quintic equation, $Q(z) = 0$.

1°. The rational invariants A, D, C, and R are calculated.

We suppose $D \neq 0$, otherwise the double root can be calculated rationally and the quintic reduced to a cubic; also that $R \neq 0$, for otherwise one root is rationally known, the linear covariant of fifth order, α_z; and α_z vanishes identically only for special types of quintics whose solutions can be obtained by elementary methods.

2°. The square root of the discriminant, $\sqrt{D}$, is adjoined.

This determines B_1, B_2, C, the known quantities in the I'-form problem.

3°. The I'-form problem is solved to obtain the 60 points $y_1 : y_2 : y_3$.

This solution is discussed in §4, account being taken of the special cases $S_1 = 0$ { $S_2 \neq 0$, $S_2 = 0$ }.

4°. One of the 60 points determines, by (2) §2, a quintic, $Q'(s)$, whose roots, $\infty, s_1, \cdots, s_4$, are projective to the roots of $Q(z) = 0$.

5°. If α_z and δ_z are the linear covariants of degrees 5 and 13 of $Q(z)$ and D its discriminant; and if α'_s, δ'_s, D' are similar comitants of $Q'(s)$, then the linear transformation

$$\frac{D \cdot \alpha_z}{\delta_z} = \frac{D' \cdot \alpha'_s}{\delta'_s} \tag{1}$$

transforms $\infty, s_1, \cdots, s_4$ into $z_0, \cdots, z_4$, the required roots of $Q(z) = 0$.

Since $R \neq 0$, α and δ are distinct, non-vanishing covariants and the transformation (1) is perfectly definite.

§7. *Remarks on the above solution of the quintic.*

The problem of the quintic involves five parameters, the given coefficients. The process sketched above (steps 1°, 2°, 4°, 5°) by which the transition is made from this five parameter problem to the I'-form problem containing two

* The calculations necessary in the above solution seem simpler than those required in the method proposed by KLEIN (*Ikosaeder*, p. 237). The most difficult one, the calculation of TT is shorter than several there indicated. In this connection it may be noted that a similar set of three degenerate curves occurs in the linear system of rational curves employed to solve the principal equation (see pp. 190–1 and p. 204, footnote).

parameters, and *vice versa*, leaves little to be desired. The full effect of the general linear transformation is obtained and the analytical processes are entirely those of the well-known theory of binary forms. But the same can not be said of step 3°, the suggested solution of the I'-form problem. This is obtained *indirectly*, being made to depend on the problem of the a's which involves the same number of parameters and a group of the same order and which is, therefore, not essentially a simpler problem. The solution of either should involve that of the other. An independent reduction of the I'-form problem to a determination of the ikosahedral irrationality is desirable. It ought to be possible to effect this, as in the other problems, by means of the rational invariant curves. Since a rational curve can admit a group of order 60 at most, it is necessarily a function of B_1, B_2, C, unsymmetric in B_1 and B_2. The totality of such curves can be thus obtained: If α_0 and β_0 are two five-valued functions of λ, invariant under the same subgroup, γ_{12}, of γ_{60}; and α_i and β_i their conjugate values, the quintic whose roots are α_i/β_i will be represented by a point y whose coördinates are functions of λ, [(6), § 2]. On the rational curve thus obtained, the group G_{60} of y and the group γ_{60} of λ are identical. One of the two simplest rational curves, B_1 itself, is thus obtained from the simplest five valued function, t. Writing

$$t_i = \epsilon^{3i}\lambda_1(\lambda_1^5 - 2\lambda_2^5) + \epsilon^{2i}\lambda_2(2\lambda_1^5 - \lambda_2^5) - 5\lambda_1^2\lambda_2^2(\epsilon^i\lambda_1^2 + \epsilon^{4i}\lambda_2^2)$$

and indicating the quadratic factors of H by the notation

$$[2+i, 3+i] = (\epsilon - \epsilon^4)(\epsilon^i\lambda_1^2 - \delta_1^2\lambda_1\lambda_2 - \epsilon^{4i}\lambda_2^2),$$

$$[4+i, 1+i] = (\epsilon^2 - \epsilon^3)(\epsilon^i\lambda_1^2 - \delta_2^2\lambda_1\lambda_2 - \epsilon^{4i}\lambda_2^2),$$

we have for the differences of the t's, the values

$$-\sqrt{5}(t_{2+i} - t_{3+i}) = [0+i, 4+i][4+i, 1+i][1+i, 0+i],$$

$$-\sqrt{5}(t_{4+i} - t_{1+i}) = [0+i, 2+i][2+i, 3+i][3+i, 0+i].$$

Substituting these differences in (6) § 2, and taking out the common factor $\pi[i, k]$, $(i, k = 1, 2; \cdots; 3, 4)$ we have the rational representation of the sextic B_1 in the same form as (6) except that $t_i - t_k$ is replaced by the quadratic $[i, k]$. The author hopes to discuss these curves more fully in a later paper.

It may be mentioned here that by formulas (4), (5), § 4, a point a is determined by five points y, since a pencil of cubics in the net of C_1, C_2, C_3, has five movable base points. The separation of the five sets of 60 conjugate points y thus determined by a set of 60 conjugate points a can be accomplished by the general "diagonal" quintic equation. For if in equations (12), § 4, we put $B_1/B_2 = (u - 5a)/20a$ and eliminate B_1, B_2, C, we have the result

$$4cu^5 - 20bu^3 + 5au - 1 = 0.$$

Here a, b, c, being given and u determined, we have the values of B_1/B_2 and from b/a^3, the value of C/B_2^3.

§ 8. *Generalizations of the two forms of the Cremona* G_{120}.

The I-form-problem and its accompanying Cremona group, as discussed in § 2, admits of an immediate generalization to apply to the equation of the nth degree. In a space of $n-3$ dimensions, S_{n-3}, a general set of $n-1$ points is chosen, the points $1, 2, \cdots, n-1$. Through these points are ∞^{n-4} rational norm-curves of order $n-3$, R; through each point, P_1, of S_{n-3} a single curve R. At P there is a definite S_{n-5} having $(n-4)$-point contact with R. This S_{n-5} and the $n-1$ points determine a pencil of spaces S_{n-4}, which pencil also includes the space having $(n-3)$-point contact with R at P. In this pencil the $(n-1)$ spaces determined by the $n-1$ points and the hyper-osculating space fix a system of projective binary n-ics. Conversely a given binary n-ic fixes a set of $n!$ points P according to the order in which $n-1$ of its roots are associated with the $n-1$ fixed points. A single curve R is picked out by requiring the parameters of the fixed points to be projective to the selected $n-1$ roots. On this curve there is a single point P whose parameter is at the same time projective to the n-th root.

The $n!$ points P determined by a binary n-ic are a set of points conjugate under a group $G_{n!}$ of Cremona point transformations. This group is generated by the collineation group of order $(n-1)!$ which permutes the $n-1$ fixed points and by the Cremona transformation of order $n-3$ which has $n-2$ of the points as a "principal basis" and the remaining point as a fixed point, a transformation of the type $x_i' = 1/x_i$. It consists of $(n-1)!$ collineations, and $(n-1)!(n-1)$ Cremona transformations of order $n-3$ with $n-2$ of the fixed points as principal points of order $n-4$. To every invariant of the binary n-ic corresponds a locus of points P for which the corresponding invariant of the n spaces S_{n-4} determined at P vanishes. The locus I is of dimensions $n-4$ and is an invariant of the group. Conversely to every invariant spread of dimensions $n-4$ corresponds an invariant of the n-ic. To the discriminant D corresponds the square of the $\frac{1}{2}(n-1)(n-2)$ spaces through $n-3$ of the fixed points.

The $n-1$ points being given in any convenient coördinate system, the equations of the invariant spreads are obtained as before by means of the typical representation of the binary n-ic with one reference point at a zero of the n-ic and the other at the linear polar of this one with regard to the remaining $n-1$. This requires a calculation of the invariants of the n-ic in the typical form. If $I_1, I_2, \cdots, I_{n-2}$, are $n-2$ independent invariant spreads, i. e., the corresponding invariants are connected by no syzygy, we can state again the I-form problem:

Given the numerical values of $I_1, \cdots, I_{n-2}$, *or of the* $n-3$ *absolute invariants formed from them, to calculate the* $n-3$ *ratios of the coördinates of one of a movable set of* $n!$ *conjugate points for which the spreads take the assigned values.*

If $\sqrt{D}$ be adjoined as an invariant curve, the order of the group reduces to $\frac{1}{2}n!$ A new set of $n-3$ absolute invariants can now be chosen and a statement of the I'-form problem made as before. The solution of the I'-form problem gives the roots of an n-ic projective to the original n-ic. In the case of n an even number the transformation by which the transition to the given n-ic is made involves certain square roots (see CLEBSCH, *Binäre Formen*, pp. 425–7).

As a special case of the above, take $n = 6$. The binary sextic has 4 independent invariants, A, B, C, and D the discriminant, of degrees 2, 4, 6, and 10 in the coefficients, and one skew invariant R of degree 15 whose square is rational in the other four. The corresponding surfaces, invariant under the Cremona G_{720}, determined by five points in space, are of orders 4, 8, 12, 20, and 30 with 2-, 4-, 6-, 12-, and 15-fold points at the five points. Some of these invariants are determined by the transformations of period 2. D is the surface of fixed points under a transformation of the type (12). R is the product of 15 quadrics each a locus of fixed points under a transformation of the type (01) (23) (45). The 45 transformations of the type (12) (34) determine 15 line pairs and 30 conic pairs as their locus of fixed points.

The parameters in the I-form problem can be chosen to be B/A^2, C/A^3, D/A^5. In the I'-form problem occurs $\sqrt{D}$, a surface of order 10. Since $\sqrt{D}$ and R are the only two surfaces whose orders are not divisible by 4 we have to take as the simplest set of parameters B/A^2, C/A^3 and $R/(\sqrt{D})^3$. The sets of 360 conjugate points then occur as the movable meets of a doubly infinite system of curves of order 96, invariant under G_{720}, and a simply infinite system of surfaces of order 30, invariant under G_{360} only.

A method of attack that is suggested in connection with this problem is to find a system of ∞^1 "homaloidal" surfaces invariant under G_{360}. Through any set of 360 conjugate points will pass a certain number of surfaces and one of this number can be singled out by the adjunction of an accessory irrationality. On the surface so obtained, which can be rationally represented by means of three homogeneous variables, the Cremona G_{360} is represented by WIMAN'S group of 360 collineations on the three variables. Thus the solution of the I'-form problem, which contains 3 parameters, would be made to depend, by means of an accessory irrationality, on the solution of the form problem associated with WIMAN'S group which depends on only two parameters. As in the case of the quintic, where the ikosahedral binary form problem is the final one, owing to the non-existence of a unary g_{60}, so in the case of the sextic, the WIMAN form-problem is final, owing to the non-existence of a binary g_{360}. Homaloidal sur-

faces of the desired sort can be obtained from the six values of the ratios of any two six-valued functions under g_{360}, i. e., any rational functions of a, b, c, d (notation of §§ 4, 5).

The G_{120}, discussed in § 1, is generalized in a very different way from that of § 2. In the plane we find geometrical analogues of the permutation groups of four and five things determined by four and five points respectively. *That the G_{24} of the four point is a collineation rather than a Cremona group, is accidental.* The analogue of the ∞^1 projectively distinct quartics is the pencil of conics through the four points. Owing to the presence of the invariant subgroup G_4 which transforms each conic into itself, a binary quartic is represented by only six members of the pencil. To extend this representation to the sextic we fix six points $p_0, \cdots, p_5$ in a space S_3. Through these six points pass ∞^3 quadric surfaces Q. A line L of the space is contained on a definite one of the Q and on it is, say, an h-generator. The six planes through L and the points $p_0, \cdots, p_5$ determine projectively a sextic. The same sextic is determined by the parameters of the six k-generators through the point p_i and is therefore independent of the originally chosen h-generator L. Thus the lines of a regulus through the six points determine a system of projective sextics. Conversely a binary sextic determines a definite regulus through the six points. For five of the roots fix a cubic curve through the five points $p_0, \cdots, p_4$ and the sixth fixes a point q_5 on this curve. The pencil of planes on the line p_5q_5 cut the cubic in two variable points whose join lies on a regulus through the six points. The only case of exception to the correspondence of sextic and regulus occurs when the sextic is projective to the parameters of the six points on the cubic curve through them. This system of sextics is represented by the ∞^2 chords of the cubic curve. The invariants of the sextic are represented by certain line complexes containing ∞^2 reguli.

Again with reference to seven given points in space, $p_0, \cdots p_6$ a line L fixes a system of projective binary septimics, and a given septimic, with ordered roots, will determine in general a definite line. Exceptions occur for the reguli through the seven points all lines of which determine the same septimics. In both the above cases the complexes of lines corresponding to invariants of the sextic and septimic can be obtained by CLEBSCH'S principle of transference. These complexes are invariant under Cremona groups of "line transformations," i. e., Cremona groups in six variables p_{ik} which have an invariant quadric. Owing to this identity among the variables the form problem of the septimic would involve only four parameters. Practically however this is not simpler than a form problem in five dimensions with five parameters. The large increase in the identities among the variables would seem to render useless further generalizations along this line by means of which octavics and nonics are represented by planes in four dimensions with references to 8 and 9 fixed points in the space.

We have in the above various methods for basing the solution of an equation of degree n, which contains n parameters, on the solution of a form problem which contains $n-3$ parameters. The underlying group of the form problem is, to be sure, a Cremona group, more complicated therefore than the collineations groups heretofore employed. But the proof of WIMAN that no collineation group, isomorphic with the even permutation group of eight things, exists in less than six dimensions shows that the octavic can only be reduced to the solution of a collineation form problem containing not less than six parameters. On the other hand, the Cremona group form problem outlined at the beginning of this paragraph contains only five parameters. And there is no reason as yet to suppose that isomorphic Cremona groups do not exist in spaces of fewer dimensions. It is thus important to investigate the occurrence or non-occurrence of Cremona groups, isomorphic with the even permutation groups of k things, in the spaces of a moderate dimension.

In the case of the quintic we have seen that the adjunction of an irrationality (ρ in § 5 or $\sqrt{A}$ of KLEIN) permits of the solution of a form problem with two parameters in terms of the solution of a form problem with only one ($z = H^3/12^3 f^5$). In connection with the form problems associated with a given equation three kinds of irrationalities can be defined: natural, accessory, and functional.

A natural irrationality is one whose adjunction permits of reducing a form problem with a certain number of parameters and a group of a certain order to another form problem with the same number of parameters but with a group of lower order.

Thus the adjunction of $\sqrt{D}$ in § 3 lowers the order of the group from 120 to 60 while the number of parameters is the same.

An accessory irrationality is one whose adjunction permits of reducing the given form problem to another whose group has the same order but which depends on a smaller number of parameters. E. g., the adjunction of ρ in § 5 reduces the a-problem to the Z-problem, both having a group of order 60, but the first depending on two parameters, the second on one.

A functional irrationality is one whose adjunction permits of reducing the given form problem to another whose group has a lower order and which contains a smaller number of parameters.

In the case of an equation of degree n, one root of the equation is such an irrationality. So also is λ, the ikosahedral irrationality, whose adjunction reduces g_{60} to g_1 and reduces the number of parameters to zero.

The plan of procedure in solving a general equation of degree n suggested by the above is:

(1) To obtain the values of the $n-3$ parameters in *any* I-form problem, whose group is $G_{n!}$.

(2) To obtain, after the adjunction of the natural irrationality $\sqrt{D}$, the $n-3$ parameters in the I'-form problem whose group is $G_{\frac{1}{2}n!}$.

(3) To reduce, by the adjunction of successive accessory irrationalities, the number of parameters as far as possible, i. e., until the group, collineation or Cremona, isomorphic with $G_{\frac{1}{2}n!}$, in the smallest number of variables, has been attained.

(4) To adjoin, as a functional irrationality, the solution of this final form problem.

(5) To obtain from this solution, by rational processes (or also by the irrationalities involved in a typical representation of the n-ic) the roots of the given equation.

The writer hopes to apply the process here sketched to the sextic in a later paper.

BALTIMORE, *February* 1, 1908.

ON THE DIFFERENTIAL EQUATIONS OF THE EQUILIBRIUM OF AN INEXTENSIBLE STRING*

BY

EDWIN BIDWELL WILSON

1. **Introduction.** In a recent communication on the problem of the rotating string or chain, I remarked the fact that of the great variety of problems on the equilibrium of strings there are few, if any, which treat cases other than those in which the field of force is rectilinear, namely parallel or central, and that consequently problems in which the string has a free end are too trivial, as far as the general question of equilibrium is concerned, to deserve mention.† The object of the present discussion is to enlarge upon that remark. It will appear in § 6 that there is a large class of cases other than rectilinear which may be explicitly integrated by quadratures.

Let it be assumed that the string lies in a plane field of force derivable from the potential V. The condition of equilibrium is then expressible by means of the calculus of variations as

$$\int \rho V ds \text{ a minimum}, \qquad \int ds \text{ a constant},$$

where ρ is the density.‡ The integral to render a minimum becomes

$$\int F ds = \int (\rho V + \lambda)\, ds \qquad (\lambda \text{ a parameter}). \tag{1}$$

This type of integral, where the integrand is the product of a function of position and the differential of arc, is of frequent occurrence in the applications of the calculus of variations. It includes the propagation of light in a medium, in which F is the index of refraction, and the brachistochrone problems, where F is the reciprocal of the velocity.

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† *The equilibrium of a heavy homogeneous chain in a uniformly rotating plane*, Annals of Mathematics, series 2, volume 9 (1908), pp. 99–115.

‡ The density may be considered as constant, although the case in which the density were a function of position (a condition rarely, if ever, realized in practice) is not excluded in the analysis; it would merely be necessary to replace F by ρF and note that the curves called equipotential in § 2 would be curves $\rho F = \text{const.}$

The differential equation of the extremals in cartesian coördinates is

$$\frac{d}{dx}\left(F\frac{y'}{\sqrt{1+y'^2}}\right)-\sqrt{1+y'^2}\frac{\partial F}{\partial y}=0 \tag{2}$$

or

$$Fy''=(F_y-F_xy')(1+y'^2). \tag{3}$$

This latter equation may be written in simpler form as

$$y''+\phi_xy'^3-\phi_yy'^2+\phi_xy'-\phi_y=0, \qquad \phi=\log F. \tag{4}$$

These equations are of the second order and third degree. Such equations have the property that their degree as well as their order is unchanged by the general point transformation

$$\bar{x}=X(x,y), \qquad \bar{y}=Y(x,y), \tag{5}$$

except that the resulting equation may be of lower degree owing to the vanishing of some of the coefficients in the transformed equation.

The question of the integrability of an equation of the second order may be viewed in several lights. From LIE's point of view, the question is as to the existence of 0, 1, 2, 3, or 8 independent infinitesimal transformations admitted by the equation.* In this direction LIE himself and his pupil TRESSE have carried on investigations in which the differential invariants figure prominently.† These methods are applicable to equations of higher degree than the third. On the other hand R. LIOUVILLE, without making use of LIE's methods but none the less by employing invariants of (5), has discussed this case of a cubic in y' in extended detail.‡ Leaving for the present the further consideration of these methods, I shall begin without introducing the theory of invariants.

If the equation $y''=\omega(x,y,y')$ is to admit the infinitesimal transformation

$$Uf=\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y} \tag{6}$$

the condition given by LIE§

$$\begin{aligned}(\eta_y-2\xi_x-3\xi_yy')\omega-\xi_{yy}y'^3+(\eta_{yy}-2\xi_{xy})y'^2+(2\eta_{xy}-\xi_{xx})y'+\eta_{xx}\\ -\xi\omega_x-\eta\omega_y-[\eta_x+(\eta_y-\xi_x)y'-\xi_yy'^2]\omega_{y'}\equiv 0\end{aligned} \tag{7}$$

* See, for example, LIE-SCHEFFERS, *Differentialgleichungen*, especially Abteilung 5.

† TRESSE, *Sur les invariants différentielles des groupes continus de transformations*, Acta Mathematica, vol. 18 (1893), pp. 1–88, and *Détermination des invariants ponctuelles de l'equation ordinaire du second ordre* $y''=\omega(x,y,y')$, Jablonowski Preisschrift (1896).

‡ *Sur quelques équations différentielles non linéaires*, Journal de l'Ecole polytechnique, vol. 57 (1887), pp. 187–250, and *Mémoire sur les invariants de certaines équations différentielles et leurs applications*, ibid., vol. 59 (1889), pp. 7–76.

§ LIE-SCHEFFERS, *Differentialgleichungen*, Theorem 35, p. 363.

must be fulfilled identically in y'. This requires the four simultaneous linear partial differential equations

$$
\begin{aligned}
\eta_{xx} - 2\xi_x\phi_y + \eta_x\phi_x + \eta_y\phi_y - \xi\phi_{xy} - \eta\phi_{yy} &= 0, \\
\xi_{yy} - 2\eta_y\phi_x + \xi_y\phi_y + \xi_x\phi_x - \xi\phi_{xx} - \eta\phi_{xy} &= 0, \\
\eta_{yy} - 2\xi_{xy} + 2\xi_y\phi_x + 3\eta_x\phi_x - \eta_y\phi_y - \xi\phi_{xy} - \eta\phi_{yy} &= 0, \\
\xi_{xx} - 2\eta_{xy} + 2\eta_x\phi_y + 3\xi_y\phi_y - \xi_x\phi_x - \xi\phi_{xx} - \eta\phi_{xy} &= 0,
\end{aligned} \tag{8}
$$

of the second order in the two independent variables x, y and the two dependent variables ξ, η to have a common solution.

It is physically obvious and readily verified on equations (8) that, if F and consequently V and ϕ are functions of one variable alone, say x, there is a common solution $\xi = 0$, $\eta = 1$ which amounts to a translation along the equipotentials $V = \text{const.}$ In like manner if F is a function of $r = \sqrt{x^2 + y^2}$ alone there is the obvious common solution $\xi = -y$, $\eta = x$ which represents a rotation about the center of force at the origin. The complete integration of these two cases seems to have been familiar to JOHN BERNOULLI;* the results are

$$
y = \int \frac{dx}{(c^2F^2 - 1)^{\frac{1}{2}}}, \qquad \theta = \int \frac{dr}{r(c^2F^2r^2 - 1)^{\frac{1}{2}}} \quad (c \text{ a constant}), \tag{9}
$$

in terms of quadratures, from which it is easy to see cases that are and cases that are not such that the form of the curve can be expressed by trigonometric, hyperbolic, or elliptic functions.

2. Transformation to equipotential curves. If the reason for the integrability by quadratures in the foregoing cases be sought, it is seen to lie in the fact that the equation (2) in the first case, and the similar equation in polar coördinates in the second case, reduces to one term and is immediately integrable to

$$
F\frac{y'}{\sqrt{1 + y'^2}} = \frac{1}{c}, \qquad F\frac{r^2\theta'}{\sqrt{1 + r^2\theta'^2}} = \frac{1}{c}, \tag{10}
$$

wherein the variables are separable. Now, inasmuch as F is a function of V alone, a transformation to curvilinear coördinates u, v, in which one set of curves is the set of equipotentials and the other set is any independent family will take (1) into the form

$$
\int F ds, \qquad ds = \sqrt{A du^2 + 2B du\, dv + C dv^2} \tag{11}
$$

where F is a function of V alone, and if the method of integration used in (10) and (9) is to be available it is necessary and sufficient that A, B, C be likewise functions of V only.

*See the numerous references in WALTON'S *Problems in Theoretical Mechanics*, third edition, Chapter 5.

Considerably to simplify the further calculation of these conditions and others that will arise it is desirable to transform from cartesian to minimal coördinates

$$(12)\quad m=x+iy,\ \ n=x-iy;\quad x=\frac{1}{2}(m+n),\ \ y=\frac{1}{2i}(m-n);\quad ds=\sqrt{dmdn}.$$

Then equations (2), (3), (4) become

$$(2')\qquad \frac{d}{dm}\left(F\frac{1}{2\sqrt{n'}}\right)-F_n\sqrt{n'}=0,$$

$$(3')\qquad Fn''=2F_m n'-2F_n n'^2,$$

$$(4')\qquad n''+\phi_n n'^2-\phi_m n'=0,\qquad \phi=\log F^2,$$

where it should be particularly noted that ϕ regarded as a function of m, n is the logarithm of the square of F regarded as a function of those variables instead of the logarithm of F. Incidentally the set of equation (8) becomes

$$(8')\qquad \begin{aligned} \eta_{mm}-\phi_m\eta_m&=0,\\ \xi_{nn}-\phi_n\xi_n&=0,\\ \eta_{nn}-2\xi_{mn}+\phi_n\eta_n-2\phi_m\xi_n+\phi_{mn}\xi+\phi_{nn}\eta&=0,\\ \xi_{mm}-2\eta_{mn}+\phi_m\xi_m-2\phi_n\eta_m+\phi_{mm}\xi+\phi_{mn}\eta&=0. \end{aligned}$$

On introducing the transformation to the equipotential curves, that is,

$$(13)\quad u=g(m,n),\quad v=h(m,n),\quad du=g_m dm+g_n dn,\quad dv=h_m dm+h_n dn,$$

where it is assumed that g is a function of $V(m,n)$, the integral to minimize is (11) where by virtue of

$$(14)\quad dm=\frac{h_n du-g_n dv}{\Delta},\quad dn=\frac{g_m dv-h_m du}{\Delta},\quad \Delta=\begin{vmatrix} g_m & g_n\\ h_m & h_n\end{vmatrix}\neq 0,\quad \int F\sqrt{dmdn},$$

the expressions for A, B, C are merely

$$(15)\qquad A=-\frac{h_m h_n}{\Delta^2},\qquad 2B=-\frac{g_m h_n+g_n h_m}{\Delta^2},\qquad C=-\frac{g_m g_n}{\Delta^2};$$

and these are, under the present hypotheses, to be functions of V and consequently of g. The conditions may be restated in better form.

It is evident that

$$\frac{2B}{A}=\frac{g_m h_n+g_n h_m}{h_m h_n}=\frac{g_m}{h_m}+\frac{g_n}{h_n}\qquad \text{is a function of } u=g(m,n),$$

$$\frac{2B}{C}=\frac{g_m h_n+g_n h_m}{g_m g_n}=\frac{h_n}{g_n}+\frac{h_m}{g_m}\qquad \text{is a function of } u=g(m,n).$$

But if the sum of two quantities and the sum of their reciprocals is known, so is their product and hence their difference and finally the quantities themselves. Thus

$$\frac{g_m}{h_m} \text{ and } \frac{g_n}{h_n} \qquad \text{are functions of } u.$$

Now A may be written in the form

$$A = -\frac{h_m h_n}{\Delta \cdot \Delta} = -\frac{1}{\Delta}\left(\frac{1}{g_n/h_m - g_n/h_n}\right),$$

which shows that Δ is itself a function of u, and hence from the original forms of A, B, C, it follows that

(16) $$h_m h_n,\ g_m g_n,\ g_m h_n,\ g_n h_m \qquad \text{are functions of } u \text{ or } g.$$

It may be noted that in the derivation of these conditions the assumption has tacitly been made that none of the four quantities g_m, g_n, h_m, h_n, which occur in denominators, vanish. In case g_m or g_n should vanish there would be no need of a transformation to equipotentials. In case h_m or h_n should vanish a similar discussion of A, B, C shows that the conditions (16) are none the less satisfied. It may further be noted that of the four conditions only three, in particular the last three, are independent. Furthermore, the function g may be replaced by V or F which are functions of g: the more general function g has been introduced to cover the cases in which some function of F or V might be simpler to treat than F or V. The results may be stated as

THEOREM 1. *The conditions that the extremals of the problem*

(1) $$\int F(x, y)\, ds, \qquad ds = \sqrt{dx^2 + dy^2}, \text{ a minimum}$$

may be found by direct quadrature by transformation to the equipotentials are that the three quantities

(16′) $$g_m g_n, \qquad g_m h_n, \qquad g_n h_m, \qquad m \text{ and } n \text{ given by (12)},$$

where g is any convenient function of F, be functions of F.

3. Further discussion of the conditions. The expression of the fact that the functional determinants of each of the three functions (16′) and g vanish is

$$\begin{vmatrix} g_{mn} g_n + g_m g_{nn} & g_m \\ g_{mn} g_n + g_m g_{nn} & g_n \end{vmatrix} = 0 \qquad \text{or} \qquad g_n^2 g_{mm} = g_m^2 g_{nn},$$

$$\begin{vmatrix} g_{mm} h_n + g_m h_{mn} & g_m \\ g_{mn} h_n + g_m h_{nn} & g_n \end{vmatrix} = 0, \qquad \begin{vmatrix} g_{mn} h_m + g_n h_{mm} & g_m \\ g_{nn} h_m + g_n h_{mn} & g_n \end{vmatrix} = 0,$$

which by the introduction of the auxiliary function $\psi = g_n/g_m$ may be reduced to the three equations

$$(17)\quad g_n^2 g_{mm} - g_m^2 g_{nn} = 0, \qquad \psi h_{mn} - h_{nn} - \psi_m h_n = 0, \qquad \psi h_{mn} - \psi^2 h_{mm} + \psi_n h_m = 0.$$

This leads to a restatement of the conditions in a modified form in the

THEOREM 2. *The conditions that the extremals of* (1) *may be found by direct quadrature by transformation to the equipotentials is that g or F or V shall satisfy the partial differential equation*

$$(17a)\qquad \frac{g_{mm}}{g_m^2} = \frac{g_{nn}}{g_n^2} \qquad \text{or} \qquad \frac{\partial}{\partial m}\left(\frac{1}{g_m}\right) = \frac{\partial}{\partial n}\left(\frac{1}{g_n}\right)$$

and that a function h may be found to satisfy the equations

$$(17b)\qquad \psi h_{mn} - h_{nn} - \psi_m h_n = 0, \qquad \psi h_{mn} - \psi^2 h_{mm} + \psi_n h_m = 0,$$

subject to the restriction that $\Delta = g_m h_n - g_n h_m \neq 0$ and with $\psi = g_n/g_m$.

It may be noted that the function ψ may really be introduced because it was seen that no restriction would result from assuming that neither g_m nor g_n vanish. It is also evident that if g is a solution of (17a), so is any function of g. Furthermore certain solutions of this equation are known, namely,

$$(18)\qquad g = \Phi(am + bn + c) \qquad \text{and} \qquad g = \Phi[(m + a)(n + b)],$$

Φ an arbitrary function, a, b, c arbitrary constants, which correspond to the cases of a force parallel to a line or acting toward a center. As there are two equations (17b), it is not to be expected that they may be satisfied unless the function ψ which occurs in the coefficients is restricted; and it is hardly probable that this restriction is no more than equivalent to (17a). It may turn out, however, that in a given example not even the necessary condition (17a) is satisfied. For instance, in the previous communication in the Annals of Mathematics the potential function was of the form $x + cy^2$, which is immediately seen to fail in satisfying this condition. The transformation to equipotentials would therefore be useless.

To bring in the conditions (17b), the first equation may be differentiated with respect to m and multiplied by ψ and then added to the derivative of the second with respect to n. The derivatives of the third order drop out and so do those of the second order if h_{mm} be substituted from the second of the equations and the result reduced by the relation $\psi\psi_m + \psi_n = 0$ which is an equivalent of (17a). The resulting equation

$$(19)\qquad \psi\psi_{mm} h_n + (2\psi\psi_m^2 - \psi_{nn}) h_m = 0$$

may be added to the two in (17b) and must hold simultaneously with them. If

ψ_{mm} be expressed in terms of g, the result after discarding factors which are not zero is

$$\psi_{mm} \propto g_n g_{m^2n} - g_m g_{mn^2}.$$

That the coefficient of h_m may be obtained by interchanging m and n in this form of ψ_{mm} is obvious from the symmetry of the original conditions (16)*. Hence (19) becomes

$$(19') \qquad (g_n g_{m^2n} - g_m g_{mn^2})(h_n - h_m) = 0.$$

This condition splits into two. Either $\psi_{mm} \propto g_n g_{m^2n} - g_m g_{mn^2} = 0$ or $h_n = h_m$. In the first case ψ may be obtained at once in the form

$$\psi = mf_1(n) + f_2(n)$$

and after substitution in $\psi\psi_m + \psi_n = 0$, which was seen to be the equivalent of (17a), gives the two equations

$$f_1^2 + f_1' = 0, \qquad f_1 f_2 + f_2' = 0.$$

If f_1 and f_2 be determined from these equations the value of ψ is

$$\psi = \frac{m+a}{n+b} = \frac{g_n}{g_m} \qquad \text{or} \qquad g = \Phi[(m+a)(n+b)].$$

Thus in this case there is nothing new: for the value of g is that already determined for central forces. In the second case where $h_n = h_m$, a reference to the original conditions (16) shows that h_n and h_m are each individually functions of g and hence g_m and g_n are functions of g. The functional detrminants in question reduce to the differential equations

$$\frac{g_{mm}}{g_m} = \frac{g_{mn}}{g_n} \qquad \text{and} \qquad \frac{g_{nm}}{g_m} = \frac{g_{nn}}{g_n}$$

or

$$\frac{\partial}{\partial m} \log g_m = \frac{\partial}{\partial m} \log g_n \qquad \text{and} \qquad \frac{\partial}{\partial n} \log g_m = \frac{\partial}{\partial n} \log g_n.$$

From the latter form it follows at once that

$$\psi = \frac{g_n}{g_m} = \text{const.} \qquad \text{or} \qquad g = \Phi(am + bn + c),$$

which again is not new, merely the case of a parallel field.

The results may therefore now be summed up in the general

THEOREM A. *The cases of central and parallel field are the only cases in which a transformation to the equipotentials and to any other system of curves will enable the integration to be carried out immediately owing to the lack of one of the variables in the integrand Fds.*

* The complete work was, however, used as a check on this statement.

It therefore appears that the quest of cases in which the discussion of a string with a free end is other than trivial must be along a less restricted direction. Before attacking this problem, I will make a digression in the following article.

4. On the integration of $p^2t - q^2r = 0$. The integration of equation (17a), although not required in the discussion of the problem here treated, offers some points of interest owing partly to the fact that it belongs to the difficult class of differential equations of the second order and of degree higher than the first, partly to the method employed. In the usual notation the equation is

$$\left(\frac{\partial z}{\partial x}\right)^2 \frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial z}{\partial y}\right)^2 \frac{\partial^2 z}{\partial x^2} \quad \text{or} \quad \psi\psi_x + \psi_y = 0 \quad \text{or} \quad \psi p + q = 0 \quad \text{if} \quad \psi = \frac{\partial z}{\partial y}\Big/\frac{\partial z}{\partial x}.$$

If CHARPIT's method be applied to $\psi\psi_x + \psi_y = 0$, the resulting system of equations is

$$\frac{dy}{-1} = \frac{dp}{p^2} = \frac{dq}{pq} = \frac{d\psi}{-p\psi - q} = \frac{dx}{-z}.$$

The integration of the first two of this system gives $p^{-1} = y + b$ and hence

$$d\psi = \frac{dx}{y+b} - \frac{\psi dy}{y+b} \qquad \text{or} \qquad dx = yd\psi + \psi dy + bd\psi$$

becomes immediately integrable with the result

$$\psi = \frac{x+a}{y+b} = \frac{\partial z}{\partial y}\Big/\frac{\partial z}{\partial x}, \quad \text{compare } z = \Phi[(x+a)(y+b)],$$

which is a first complete integral and corresponds precisely with the cases of a central force in the problem above. The first general integral would be obtained by eliminating b from the two equations

$$\psi = \frac{x + \theta(b)}{y+b}, \qquad \frac{1}{\psi} = \theta'(b).$$

On the other hand the integration of the second two equations of the given system gives $bp - aq = 0$ and hence

$$\psi = \frac{b}{a} = \frac{\partial z}{\partial y}\Big/\frac{\partial z}{\partial x}, \quad \text{compare } z = \Phi(ax + by + c),$$

which again is a first complete integral and corresponds to the case of parallel forces. The first general integral here cannot be found by elimination.

The elimination is not readily carried out in case θ is an arbitrary and not an assigned function. On the other hand the first general integral of the equation may be found by interchanging the dependent variable ψ with one of the vari-

ables x, y by means of the well-known formula

$$\left(\frac{\partial\psi}{\partial x}\right)_y\left(\frac{\partial x}{\partial y}\right)_\psi\left(\frac{\partial y}{\partial\psi}\right)_x=-1; \qquad \text{whence} \qquad \psi=-\frac{\psi_y}{\psi_x}=\left(\frac{\partial x}{\partial y}\right)_\psi,$$

and

$$x=y\psi+f_1(-\psi)=y\frac{z_y}{z_x}+f_1\left(-\frac{z_y}{z_x}\right)=-y\left(\frac{\partial x}{\partial y}\right)_z+f_1\left(\frac{\partial x}{\partial y}\right)_z.$$

If λ be introduced to represent $(\partial x/\partial y)_z$, the final solution of the equation is found on eliminating λ from the two equations

$$x+\lambda y=f_1(\lambda), \qquad \sqrt{\lambda}y+f_2(z)=\int\frac{f_1'(\lambda)d\lambda}{2\sqrt{\lambda}}.$$

There appears to be no way of obtaining the general solution in a form in which no elimination is necessary.

5. On the reduction of the equation of extremals to the type $y''=0$. If the equation which determines the extremals is to admit an infinitesimal transformation, the set of four equations (8) or (8′) must have a common solution for ξ and η. The method of determining the conditions for a common solution is stated very clearly by TRESSE.* In the first place the given equations should be written in a canonical form by solving for certain of the derivatives of highest order, here the second; then the equations should be differentiated until there are enough equations to solve for all the derivatives of the resulting highest order. In this case one differentiation will suffice to allow a solution for the derivatives of the third order.

$$\begin{aligned}
&\eta_{m^3}=\phi_m\eta_{mm}+\phi_{mm}\eta_m,\\
&\eta_{m^2n}=\phi_m\eta_{mn}+\phi_{mn}\eta_m,\\
&\eta_{mn^2}=-\phi_n\eta_{mn}-\phi_{nn}\eta_m+\phi_{mn}\eta_n+\phi_{mn}\xi_m+\phi_{mn^2}\eta+\phi_{m^2n}\xi,\\
&\eta_{n^3}=(\phi_n^2-2\phi_{nn})\eta_n+3\phi_{mn}\xi_n+(\phi_n\phi_{nn}-\phi_{n^2})\eta+(\phi_n\phi_{mn}-\phi_{mn^2})\xi,\\
&\xi_{m^3}=3\phi_{mn}\eta_m+(\phi_{m^2}-2\phi_{mn})\xi_m+(\phi_m\phi_{mn}-\phi_{m^2n})\eta+(\phi_m\phi_{mn}-\phi_{m^3})\xi.\\
&\xi_{m^2n}=-\phi_m\xi_{mn}+\phi_{mn}\eta_n-\phi_{mn}\xi_n+\phi_{mn}\xi_m+\phi_{mn^2}\eta+\phi_{m^2n}\xi,\\
&\xi_{mn^2}=\phi_n\xi_{mn}+\phi_{mn}\xi_n,\\
&\xi_{n^3}=\phi_n\xi_{n^2}+\phi_{n^2}\xi_n.
\end{aligned} \tag{20}$$

Next the conditions of integrability

$$\eta_{m^2\cdot n}=\eta_{m^2n\cdot m}, \quad \eta_{m^2n\cdot n}=\eta_{mn^2\cdot m}, \quad \cdots, \quad \xi_{mn^2\cdot n}=\xi_{n^2\cdot m}$$

must be expressed. In this case there are six such conditions. of which two

*Acta Mathematica, vol. 18 (1894), p. 9.

vanish identically and four are equal in pairs. There remain then

$$(21)\qquad \begin{aligned} &A(2\xi_m+\eta_n)-B\eta_m+A_n\eta+A_m\xi=0, \qquad A=\phi_m\phi_{mn}-\phi_{m^2n},\\ &B(2\eta_n+\xi_m)-A\xi_n+B_n\eta+B_m\xi=0, \qquad B=\phi_n\phi_{mn}-\phi_{mn^2}. \end{aligned}$$

These equations are to be joined with (8′) and (20) and further conditions of integrability are to be sought.

Before taking this matter up, it may be noted that if (21) are satisfied identically, the conditions of integrability are certainly fulfilled and no further work is necessary. This will be the case if $A=B=0$. A reference to the work of R. LIOUVILLE* will show that these conditions are the same as his $L_1=L_2=0$, which are the necessary and sufficient conditions that the equation (4′) should be reducible by a point transformation to the form $y''=0$. The conditions are also identical with TRESSE's $H=0$ which ensures the same possibility of reduction to $y''=0$.† It should be noted that $A=B=0$ if $\phi_{mn}=0$, that is, if ϕ or $\log F$ satisfies Laplace's equation. It is therefore possible to state

THEOREM 3. *If* $\log F$ *satisfies Laplace's equation* $\Delta \log F=0$, *that is, if*

$$(22)\qquad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log F=\frac{\partial^2}{\partial m\,\partial n}\log F=0, \qquad F=f_1(m)f_2(n),$$

the extremals may be transformed by a suitable transformation $\bar{x}=X(x,y)$, $\bar{y}=Y(x,y)$ *into the* ∞^2 *straight lines of the plane.*

If $\phi_{mn}\neq 0$ while $A=B=0$, the equations $A=0$ and $B=0$ may be put in the form

$$\frac{\partial}{\partial m}\log\phi_{mn}=\frac{\partial}{\partial m}\phi \qquad \text{and} \qquad \frac{\partial}{\partial n}\log\phi_{mn}=\frac{\partial}{\partial n}\phi.$$

From this it is an immediate consequence that

$$(23)\qquad \phi_{mn}=\kappa e^{\phi}, \qquad \frac{\partial^2}{\partial m\,\partial n}\log F^2=\kappa F^2 \qquad (\kappa \text{ a constant}),$$

and the integral of this equation may be written in the form ‡

$$(24)\qquad F=\gamma\frac{\sqrt{f_1'(m)f_2'(n)}}{f_1(m)+f_2(n)} \qquad \text{where} \qquad \gamma^2\kappa=2.$$

This result cannot be regarded as a general case under which (22) may fall, although the differential equation (23) from which it is derived may be regarded as the general case of which LAPLACE's is a special case obtained by setting $\kappa=0$. Both results may be included, however, in the general

* An immediate consequence of the work on pp. 218–219 of the first memoir cited.

† *Jablonowski Preisschrift*, l. c., p. 56.

‡ JORDAN, *Cours d'analyse*, vol. 3, p. 360.

THEOREM *B*. *The necessary and sufficient conditions that the extremals of* $\int Fds$ *be defined by a differential equation reducible by a transformation* $\bar{x} = X(x, y)$, $\bar{y} = X(x, y)$ *to the form* $y'' = 0$ *are that*

$$\frac{\partial^2}{\partial m \partial n} \log F^2 = \kappa F^2, \tag{23}$$

where κ *is any constant (including* $\kappa = 0$*); or that*

$$F = \gamma \frac{\sqrt{f_1'(m) f_2'(n)}}{f_1(m) + f_2(n)}, \quad \kappa \neq 0 \quad \text{or} \quad F = f_1(m) f_2(n), \quad \kappa = 0, \tag{24'}$$

where f_1 *and* f_2 *are arbitrary functions of* m *and* n *respectively.*

6. The case of Laplace's equation. If $\phi_{mn} = 0$, the solution for ϕ may be written in the form of the sum of two functions $g(m)$, $h(n)$

$$\phi = g(m) + h(n) + a + b, \tag{25}$$

where the constants have been added for convenience inasmuch as ϕ enters in the equation (4') of the extremals only through its derivatives. The set of equations (8') which determine ξ and η then becomes

$$\begin{gathered} \eta_{mm} = g'(m)\eta_m, \qquad \xi_{nn} = h'(n)\xi_n, \\ \eta_{nn} - 2\xi_{mn} + h'\eta_n - 2g'\xi_n + h''\eta = 0, \\ \xi_{mm} - 2\eta_{mn} + g'\xi_m - 2h'\eta_m + g''\xi = 0. \end{gathered} \tag{26}$$

The first two of these equations may be integrated completely in the form

$$\eta = N \int e^g dm + \bar{N}, \qquad \xi = M \int e^h dn + \bar{M}, \tag{27}$$

where M and $\bar{M}$ are functions of m only and N and $\bar{N}$ functions of n only. If these values be substituted in the third equation of the set, the result may be arranged in the following manner:

$$e^{-h}(N'' + h'N' + h''N) \int e^g dm + e^{-h}(\bar{N}'' + h'\bar{N}' + h''\bar{N}) = 2(M' + g'M), \tag{28}$$

with a similar equation from the fourth of the set.

As (28) is an identity for all values of m and n and as $\int e^g dm$ and e^{-h} may without loss of generality be considered as not identically constant, it follows that the three equations

$$\begin{gathered} M' + g'M = 0, \\ N'' + h'N' + h''N = 0, \\ \bar{N}'' + h'\bar{N}' + h''\bar{N} = 0, \end{gathered} \tag{29}$$

must each be true, with three similar equations from the fourth equation of (26).

The first equation with its analogue yield the solutions

$$M = ce^{-g}, \qquad N = ke^{-f} \tag{30}$$

for M and N. To determine $\overline{M}$ and $\overline{N}$ there remain the equations

$$\overline{M}'' + g'\overline{M}' + g''\overline{M} = 0, \qquad \overline{N}'' + h'\overline{N}' + h''\overline{N} = 0. \tag{29'}$$

If it be noted that e^{-g} and e^{-f} respectively are particular solutions of these equations, the general solutions may be found as

$$\overline{M} = e^{-g}\left(c_1 \int e^{g}\, dm + c_2\right), \qquad \overline{N} = e^{-h}\left(k_1 \int e^{h}\, dn + k_2\right). \tag{30'}$$

On substituting these values of M, $\overline{M}$, N, $\overline{N}$ the expressions for ξ and η are

$$\begin{aligned} \xi e^{g} &= c \int e^{h}\, dn + c_1 \int e^{g}\, dm + c_2, \\ \eta e^{h} &= k \int e^{g}\, dm + k_1 \int e^{h}\, dn + k_2. \end{aligned} \tag{31}$$

Thus in case $\phi_{mn} = 0$, the exact form of the coefficients ξ, η of Uf may be found. The group is clearly one of eight parameters. The path curves are given by the differential equations

$$\frac{dm}{\xi} = \frac{dn}{\eta} = \frac{e^{g}\, dm}{c\int e^{h}\, dn + c_1 \int e^{g}\, dm + c_2} = \frac{e^{h}\, dn}{k\int e^{g}\, dm + k_1 \int e^{h}\, dn + k_2},$$

which may be written in the form

$$\frac{k}{2} G^2 + k_1 GH + k_2 G = \frac{c}{2} H^2 + c_1 GH + c_2 H + \text{const.}, \tag{32}$$

where the notation $G = \int e^{g}\, dm$, $H = \int e^{h}\, dn$ has been introduced. Moreover in this case the integral of (4') may be found directly. For

$$\frac{n''}{n'} + h'n' - g' = 0 \qquad \text{or} \qquad d\log n' + h'\, dn - g'\, dm = 0,$$

$$\log n' + h - g = C, \qquad n' = Ce^{g-h},$$

and

$$e^{h}\, dn = Ce^{g}\, dm \qquad \text{or} \qquad H = CG + K,$$

where C and K are constants. The value of F^2 is $F^2 = Ae^{g}e^{h}$.

Now that the computations have been completed it will be advantageous to introduce the new functions $\bar{g} = e^{g}$, $\bar{h} = e^{h}$ in terms of which to state the results in

THEOREM 4. *The necessary and sufficient conditions that the differential equation* (4') *of the extremals be an exact differential equation is that the function F be defined by $F^2 = A\bar{g}\bar{h}$, in which case the solution of the equation is*

$$C_1 H + C_2 G = K, \qquad G = \int \bar{g}\, dn, \qquad H = \int \bar{h}\, dn, \tag{33}$$

and the infinitesimal transformations which the equation admits are

$$(31') \qquad \bar{\xi y} = cH + c_1 G + c_2, \qquad \eta\bar{h} = kG + k_1 H + k_2.$$

To obtain real solutions ad libitum the conjugate harmonic functions R and S may be introduced so that

$$\bar{y} = R + Si, \qquad \bar{h} = R - Si, \qquad F^2 = A(R^2 + S^2).$$

Formulated as a theorem, the result may be stated in

THEOREM 5. *If $F^2 = A(R^2 + S^2)$, where R and S are conjugate harmonic functions, the equation (4) of the extremals is integrable as*

$$(33') \qquad C_1 \int (Rdx - Sdy) + C_2 \int (Rdy + Sdx) = K.*$$

7. Concerning other cases. The next case to consider would be that of equation (23) in which $\kappa \neq 0$. Here it appears impossible to carry through the determination of the infinitesimal transformations and of the solution of the differential equation of the extremals in terms of the functions f_1 and f_2 which enter in F. Both LIE and R. LIOUVILLE have shown that in case any differential equation is reducible to the form $y'' = 0$, the solution of the given equation may be made to depend upon the solution of a linear differential equation of the third order.† The specialization which their results undergo when applied to the particular equation (4′) does not appear to be sufficient to render the equation of the third order integrable in terms of quadratures. For further details concerning this case reference is therefore made to the original sources.

Thus far the discussion has been wholly on the assumption that the given equation (4′) admits a group of eight parameters. To derive the conditions, which will be in the shape of partial differential equations, that ϕ must satisfy in order that (4′) shall admit a group of one or two or three parameters is an exceedingly tedious task at computation, whether pursued by the differentiation of (21) and the comparison with (8′) to obtain further conditions of integrability or attacked by the more systematic methods of differential invariants as indicated by TRESSE or R. LIOUVILLE.‡ Moreover, the partial differential equa-

* That this formula satisfies (3) or (4) may be seen at once on eliminating C_1, C_2, K by differentiation are on substituting $R_x = S_y$, $R_y = -S_x$, the conditions for conjugate harmonic functions.

† LIE, in a memoir quoted as Norw. Arch., 8 (1883), p. 372, in the Mathematical Encyclopedia and as Archives Norwégiennes, 8 (1883), by TRESSE. I have not had access to this reference. LIOUVILLE, in the first memoir cited. LIOUVILLE's reduction is made under certain restrictive hypotheses, see p. 240, which are ample, however, to cover the case of equation (4′).

‡ TRESSE in both memoirs, especially the latter, where the restriction of the third degree is not imposed. See references on p. 426, viz. Acta Mathematica, p. 76, and *Jablonowski Preisschrift*, pp. 60–84. LIOUVILLE, in both memoirs, especially the second. LIOUVILLE does not determine the special conditions relative to groups of two or three parameters as distinct from groups of one parameter, but this classification can be imposed if desired.

tions which would be obtained would be of so high an order and degree as to be, in all probability, of no very great value or interest when obtained. It appears therefore that although TRESSE has obtained in an elegant manner and stated in concise form the necessary and sufficient conditions that a given equation of the second order shall admit a group of one or two or three parameters, yet for practical purposes there is no method of telling whether an assigned equation does or does not admit a group.

In simple cases it is frequently possible to set up and actually solve the set of simultaneous partial differential equations for ξ and η which arise from the application of LIE's condition (7) to a given equation; and in such cases as I have tried it appears that this method of actual solution, which will give the values of ξ and η if there are any, is no longer, if as long, as the methods of differential invariants. For instance, there are five equations,

$$
\begin{aligned}
&y'' = 6y^2 + x, \\
&y'' = 2y^3 + xy + \alpha, \\
&y'' = \frac{y'^2}{y} + e^x(1 - y^2), \\
&y' = \frac{y'^2}{y} + e^x(\alpha y^2 + 1) - e^{2x}y^3, \\
&y'' = \frac{y'^2}{y} + e^x(\alpha y^2 + \beta) + e^{2x}\left(\frac{1}{y} - y^3\right),
\end{aligned}
\tag{34}
$$

of particular interest owing to the fact shown by PAINLEVÉ, that they are the only equations of the second order $y'' = R(y', y, x)$, where R is rational in y', algebraic in y, and analytic in x, which have fixed critical points and which define functions that are uniform over the entire complex plane and are *new* transcendents. *

The substitution of $\omega = 6y^2 + x$ in (7) gives the four equations

$$
\begin{gathered}
\xi_y = 0, \qquad \eta_{yy} - 2\xi_{xy} = 0, \\
2\eta_{xy} - \xi_{xx} - (18y^2 + 3x)\xi_y = 0, \\
\eta_{xx} + (\eta_y - 2\xi_x)(6y^2 + x) - \xi - 12\eta y = 0.
\end{gathered}
$$

The first two may be solved completely for ξ and η to give the results

$$
\xi = f(x)y + g(x), \qquad \eta = f'(x)y^2 + h(x)y + k(x).
$$

The substitution of these values in the third equation yields

$$
4f'''(x)y + 2h'(x) - f''(x)y - g''(x) - (18y^2 + 3x)f(x) = 0
$$

* PAINLEVÉ, *Sur les equations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme*, Acta Mathematica, vol. 25 (1902), pp. 1–85. See § 11, p. 13.

or

$$f(x)=0, \qquad 2h'(x)-g''(x)=0.$$

The additional equations obtainable from the fourth equation are

$$h(x)+2g'(x)=0, \quad k''(x)-g(x)+x[h(x)-2g'(x)]=0, \quad h''(x)-12k(x)=0,$$

and a combination of these with the foregoing shows that $f=g=h=k=0$. Hence $\xi=0$ and $\eta=0$; and the first equation of (34) has no group. The procedure for the second equation of (34) is quite similar.

The result of substitution from PAINLEVÉ's third equation in (7) is

$$\xi_{yy}+\frac{1}{y}\xi_y=0, \qquad \xi_y=\frac{f(x)}{y}, \qquad \xi=f(x)\log y+g(x),$$

$$\eta_{yy}-\frac{2f'(x)}{y}-\left(\frac{\eta}{y}\right)_y=0, \qquad \eta_y-2f'(x)\log y-\frac{\eta}{y}=h(x),$$

$$\eta=yf'(x)(\log y)^2+yh(x)\log y+yk(x),$$

from setting the coefficients of y^3 and y^2 equal to zero; and from the rest

$$-3\xi_y e^x(1-y^2)+2\eta_{xy}-\xi_{xx}-2\frac{\eta_x}{y}=0,$$

$$(\eta_y-2\xi_x)e^x(1-y^2)+\eta_{xx}-\xi e^x(1-y^2)+2\eta y e^x=0.$$

The substitution of the values of ξ and η in these equations and the identification of the coefficient of each power of y to zero shows that here again $\xi=\eta=0$. And the two remaining cases are similar; in fact the forms of ξ and η as determined from the coefficients of y^3 and y^2 are the same. Hence

THEOREM *C*. *Painlevé's five equations admit no groups.*

This result, which was merely derived incidentally to showing how the direct determination of the problem without the use of invariants is frequently possible, is interesting, though certainly not unexpected. It may be noted, however, that this absence of a group is not a characteristic of equations which define new uniform transcendents in PAINLEVÉ's sense. Such equations may admit relatively simple groups.* Moreover, the groups which he employs are algebraic in y, and it is conceivable that even though his canonical equations admitted no such group, they might still admit some group transcendental in y. Theorem *C* states that the five types of the second order do not.

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BOSTON, MASS., *March*, 1908.

* PAINLEVÉ, loc. cit., § 55, pp. 80–81.

THE PROPERTIES OF CURVES IN SPACE WHICH MINIMIZE A DEFINITE INTEGRAL*

BY

MAX MASON AND GILBERT AMÈS BLISS

While the theory of the calculus of variations from the stand-point of WEIERSTRASS has been treated extensively for problems in the plane by KNESER, BOLZA and many other writers, comparatively little seems to have been published concerning the extension of the Weierstrassian theory to problems in space. It is the purpose of the present paper to discuss from as geometrical a stand-point as possible the arcs

$$x = x(t), \qquad y = y(t), \qquad z = z(t) \qquad (t_0 \leqq t \leqq t_1), \tag{1}$$

which minimize or maximize an integral of the form

$$J = \int_{t_0}^{t_1} F(x, y, z, x', y', z')\, dt$$

involving three dependent variables, where F satisfies the homogeneity condition

$$F(x, y, z, \kappa x', \kappa y', \kappa z') = \kappa F(x, y, z, x', y', z') \qquad (\kappa > 0). \tag{2}$$

The end-points of the minimizing curve may be fixed, or one of them may vary on a given curve or surface.

The papers hitherto published with regard to problems of this type have for the most part dealt with the derivation of necessary conditions by means of the first and second variations.† In the following discussion the necessary conditions which an arc C must satisfy in order to minimize or maximize the integral are derived without the use of the second variation. Sufficient conditions which insure a minimum or a maximum are derived with the help of the Hilbert invariant integral. No discussion of the problem which might be called

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† In the dissertation of GERNET, the integral is taken in the form

$$\int f\left(x, y, z, \frac{dy}{dx}, \frac{dz}{dx}\right) dx,$$

and necessary conditions are derived without the use of the second variation. See N. GERNET, *Untersuchungen zur Variationsrechnung*, Dissertation, Göttingen, 1902.

complete, so far as is known to the writers, has hitherto been published, and but one writer* has discussed the problem in parametric representation. The study of the sufficient conditions for fixed and for variable end-points seems to be entirely new.

The first six sections contain the theory of the problem when both end-points are fixed, or when one is allowed to vary on a surface. The case when one end-point is allowed to vary on a curve is treated in the last two sections.

§ 1. *The Euler equations.*

In order to carry through the methods applied in this paper it will be assumed that the function F is of class C''' for all points (x, y, z) in a region R of space, and for all sets of values of (x', y', z') different from $(0, 0, 0)$.† The functions of the form (1) defining arcs along which the value of the integral is to be taken will be supposed to be of class C' and such that for every value of t in the interval $t_0 \leqq t \leqq t_1$ the values x, y, z, x', y', z' are of the kind just described. When an end-point is allowed to vary on a curve or on a surface, the functions defining the curve or surface are supposed to be of class C'.

A number of identities can be derived from (2) which will be useful in this and succeeding sections. By differentiating (2) with respect to x', it is seen that

$$F_{x'}(x, y, z, \kappa x', \kappa y', \kappa z') = F_{x'}(x, y, z, x', y', z'), \tag{3}$$

and in general it follows that a derivative of F with respect to x', y', z' is homogeneous of order $1 - n$, where n is the order of the derivative. By differentiating (2) with respect to κ and then letting $\kappa = 1$, we obtain the equation

$$x'F_{x'} + y'F_{y'} + z'F_{z'} = F. \tag{4}$$

From this equation follow immediately by differentiation

$$\begin{aligned} x'F_{x'x'} + y'F_{x'y'} + z'F_{x'z'} &= 0, \\ x'F_{y'x'} + y'F_{y'y'} + z'F_{y'z'} &= 0, \\ x'F_{z'x'} + y'F_{z'y'} + z'F_{z'z'} &= 0. \end{aligned} \tag{5}$$

From the latter set it is seen that the co-factors A_{ik} of the elements of the determinant of the first members satisfy the equations

$$x' : y' : z' = A_{11} : A_{12} : A_{13} = A_{21} : A_{22} : A_{23} = A_{31} : A_{32} : A_{33}.$$

By multiplying these three proportions by x', y', z', respectively, it follows that there exists a function F_1 such that

* Von Escherich, *Die zweite Variation der einfachen Integrale*, Sitzungsberichte der Akademie der Wissenschaften in Wien, vol. CX (1901), p. 1355.

† A function is said to be of class $C^{(k)}$ if it is continuous and has continuous derivatives up to and including those of order k.

$$
\begin{aligned}
x'^2F_1 &= A_{11} = F_{y'y'}F_{z'z'} - F^2_{y'z'},\\
y'^2F_1 &= A_{22} = F_{z'z'}F_{x'x'} - F^2_{z'x'},\\
z'^2F_1 &= A_{33} = F_{x'x'}F_{y'y'} - F^2_{x'y'},\\
y'z'F_1 &= A_{23} = F_{x'y'}F_{x'z'} - F_{x'x'}F_{y'z'},\\
z'x'F_1 &= A_{31} = F_{y'z'}F_{y'x'} - F_{y'y'}F_{z'x'},\\
x'y'F_1 &= A_{12} = F_{z'x'}F_{z'y'} - F_{z'z'}F_{x'y'}.
\end{aligned} \tag{6}
$$

It follows also that

$$
\begin{vmatrix}
F_{x'x'} & F_{x'y'} & F_{x'z'} & x'\\
F_{y'x'} & F_{y'y'} & F_{y'z'} & y'\\
F_{z'x'} & F_{z'y'} & F_{z'z'} & z'\\
x' & y' & z' & 0
\end{vmatrix} = -F_1(x'^2 + y'^2 + z'^2)^2, \tag{7}
$$

and the minors of this determinant of the type

$$
\begin{vmatrix}
F_{x'x'} & F_{x'y'} & F_{x'z'}\\
F_{y'x'} & F_{y'y'} & F_{y'z'}\\
x' & y' & z'
\end{vmatrix} = F_1 z'(x'^2 + y'^2 + z'^2) \tag{8}
$$

are similarly expressible in terms of F_1 and cannot all vanish simultaneously when F_1 is different from zero, since one at least of x', y', z' is always different from zero.

Suppose now that an arc C is at hand which minimizes the integral J. Then by a method quite similar to that used by WHITTEMORE,* it can be shown that along the arc C the three equations

$$
F_{x'} = \int_{t_0}^{t} F_x\,dt + a, \qquad F_{y'} = \int_{t_0}^{t} F_y\,dt + b, \qquad F_{z'} = \int_{t_0}^{t} F_z\,dt + c, \tag{9}
$$

must be satisfied, where a, b, c are constants. It is evident that a, b, c must have the same values all along C since x, y, z, x', y', z' are continuous functions of t. Let us now consider the second members of equations (9) as functions of t, and the first members as functions of x', y', z', and t, the latter occurring in the functions $x(t)$, $y(t)$, $z(t)$. The parameter t may be chosen so as to represent the length of arc along C, and then if we assume $F_1 \neq 0$ in the expression (8), it follows from the theory of implicit functions that two of the equations (9) can always be solved with

$$
x'^2 + y'^2 + z'^2 = 1,
$$

* *Lagrange's Equation in the Calculus of Variations and the Extension of a Theorem of Erdmann*, Annals of Mathematics, 2d series, vol. 2 (1900–1901), p. 130.

for x', y', z' in terms of t in the neighborhood of any set of values t, x', y', z' on the arc C. The resulting functions $x'(t)$, $y'(t)$, $z'(t)$, have continuous derivatives with respect to t, and on account of the uniqueness of such solutions must coincide with the values of x', y', z' along C. By differentiating equations (9) we have the following theorem:

If an arc C is of class C' and minimizes the integral J, the parameter can always be so chosen that the representation of C is of class C'' and satisfies the three Euler differential equations

$$(10) \qquad \frac{dF_{x'}}{dt} - F_x = 0, \qquad \frac{dF_{y'}}{dt} - F_y = 0, \qquad \frac{dF_{z'}}{dt} - F_z = 0.$$

These three equations are not independent, for by multiplying their first members by x', y', z', respectively, and adding, one finds the expression

$$\frac{d}{dt}(x'F_{x'} + y'F_{y'} + z'F_{z'}) - (x''F_{x'} + y''F_{y'} + z''F_{z'} + x'F_x + y'F_y + z'F_z),$$

which with the aid of equation (4) reduces to zero. It is evident then that any one of the equations (10) is a result of the other two.

The nature of the solutions of Euler's equations may be shown by reducing the equations to a system of differential equations in canonical form. It may be assumed without loss of generality that the length of arc is the parameter, so that

$$(11) \qquad x'x'' + y'y'' + z'z'' = 0.$$

The three equations (10), with (11), define x'', y'', z'' as functions of x, y, z, x', y', z'. This may be shown most symmetrically, perhaps, by making use of an auxiliary variable u. The equations

$$(12) \quad \frac{dF_{x'}}{dt} - F_x + x'u = 0, \qquad \frac{dF_{y'}}{dt} - F_y + y'u = 0, \qquad \frac{dF_{z'}}{dt} - F_z + z'u = 0,$$

and (11), are linear in x'', y'', z'', u, with the determinant (7) as their functional determinant. The expressions

$$(13) \quad x'' = A(x, y, z, x', y', z'), \quad y'' = B(x, y, z, x', y', z'), \quad z'' = C(x, y, z, x', y', z'),$$

found by solving them are solutions also of equations (10), for by multiplying equations (12) by x', y', z', respectively, it is seen that the corresponding value of u must vanish identically. We assume now $F_1 \neq 0$ for all values of x, y, z, x', y', z' for which it is defined. Then the determinant (7) is different from zero and the second members of equations (13) are of class C' on account of the original assumptions on the function F.

The existence theorems for differential equations in the form (13) justify the following statement, the curves defined by equations (10) being called as usual extremals:

If the problem under consideration is a regular one, that is, if the function F_1 is different from zero for all values of its arguments, then through any point (x_0, y_0, z_0) of the region R there passes one and only one extremal in a given direction (x_0', y_0', z_0'). These extremals extend from boundary to boundary of R and are defined analytically by equations of the form

$$\begin{aligned} x &= \phi(s, x_0, y_0, z_0, x_0', y_0', z_0'), \\ y &= \psi(s, x_0, y_0, z_0, x_0', y_0', z_0'), \\ z &= \chi(s, x_0, y_0, z_0, x_0', y_0', z_0'), \end{aligned} \tag{14}$$

where

$$x_0'^2 + y_0'^2 + z_0'^2 = 1, \tag{15}$$

and for $s = 0$ the following initial conditions are satisfied,

$$\begin{aligned} &\phi(0, x_0, y_0, z_0, x_0', y_0', z_0') = x_0, \qquad \phi_s(0, x_0, y_0, z_0, x_0', y_0', z_0') = x_0', \\ &\psi(0, x_0, y_0, z_0, x_0', y_0', z_0') = y_0, \qquad \psi_s(0, x_0, y_0, z_0, x_0', y_0', z_0') = y_0', \\ &\chi(0, x_0, y_0, z_0, x_0', y_0', z_0') = z_0, \qquad \chi_s(0, x_0, y_0, z_0, x_0', y_0', z_0') = z_0'. \end{aligned} \tag{16}$$

The functions ϕ, ϕ_s are of class C' for all values of their arguments defining (x, y, z)-points in the region R, and similar statements hold true for ψ and χ.

§ 2. *Conditions of transversality.*

If it is desired to minimize the integral J with respect to curves having one end-point $1(x_1, y_1, z_1)$ fixed, while the other $0(x_0, y_0, z_0)$ may vary on a fixed curve or surface, a second necessary condition upon the direction of the minimizing curve C at the point 0, corresponding to the well-known transversality condition in the plane, must be satisfied. The case where the end-point is variable on a fixed curve L will first be considered.

Let the equations of the curve L be

$$(L) \qquad x = x(u), \qquad y = y(u), \qquad z = z(u),$$

the functions being of class C' as specified in § 1. The minimizing extremal C can be imbedded in a one-parameter family of curves

$$(V) \qquad x = f(t, u), \qquad y = g(t, u), \qquad z = h(t, u),$$

which intersect the curve L for $t = t_0$, pass through the point 1 for $t = t_1$, and contain the extremal C for the particular parameter value u_0. Analytically these properties are expressed by the following identities in u:

$$\begin{aligned} x(u) &= f(t_0, u), \qquad y(u) = g(t_0, u), \qquad z(u) = h(t_0, u), \\ x_1 &= f(t_1, u), \qquad y_1 = g(t_1, u), \qquad z_1 = h(t_1, u). \end{aligned}$$

The value of the integral J taken along a member V of this family of curves from the point of intersection 0 of V with L to the fixed point 1, is a function of the parameter u,

$$J(u) = \int^{t_1} F(f, g, h, f_t, g_t, h_t)\, dt.$$

If C minimizes the integral, the function $J(u)$ has a minimum for $u = u_0$. Its derivative is

$$\frac{dJ}{du} = F_{x'} f_u + F_{y'} g_u + F_{z'} h_u \Big]_{t_0}^{t_1} - \int_{t_0}^{t_1} (Pf_u + Qg_u + Rh_u)\, dt,$$

where P, Q, R stand for the first members of the Euler equations (10). Since C is an extremal and all of the curves V pass through the point 1 for $t = t_1$, it follows that

$$\left[\frac{dJ}{du}\right]_{u=u_0} = -\left[F_{x'} f_u + F_{y'} g_u + F_{z'} h_u\right]^{t=t_0},$$

where the arguments of the derivatives of F are the values of x, y, z, x', y', z' on the extremal C at the point 0. A second necessary condition for a minimum is therefore the following:

The direction of the extremal C at its point of intersection 0 with the fixed curve L must satisfy the equation

$$F_{x'} x_u + F_{y'} y_u + F_{z'} z_u = 0, \tag{17}$$

where the arguments of the derivatives of F are the values of x, y, z, x', y', z' on the curve C, and x_u, y_u, z_u define the direction of the curve L.

From this condition one may easily derive the equations which must hold when the end-point is allowed to vary on a surface S,

$$(S) \qquad x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v).$$

For if C minimizes J with respect to all curves joining the point 1 to the surface S, it must certainly minimize J with respect to all curves which join 1 with either of the two parameter lines of the surface through the point 0. The transversality condition for the end-point varying on a surface is therefore the following:

The direction of the extremal C at its point of intersection 0 with the surface S must satisfy the equations

$$F_{x'} x_u + F_{y'} y_u + F_{z'} z_u = 0, \qquad F_{x'} x_v + F_{y'} y_v + F_{z'} z_v = 0, \tag{18}$$

where the arguments of the derivatives of F are the values x, y, z, x', y', z' on the curve C, and x_u, y_u, z_u; x_v, y_v, z_v define the directions of the parameter lines of the surface S.

It is evident that if these equations are satisfied, any curve of the surface which passes through the point 0 will also be transversal to C at 0, since the direction cosines of any curve on the surface are proportional to expressions of the form

$$x_u u' + x_v v', \quad y_u u' + y_v v', \quad z_u u' + z_v v'.$$

Furthermore the derivatives $F_{x'}$, $F_{y'}$, $F_{z'}$ are proportional to the direction cosines of the normal to the surface, as is easily seen from equations (18).

§ 3. *The family of extremals through a given point, or to which a given surface is transversal.*

Before proceeding to derive the equations of a family of extremals through a given point, let us consider for a moment the quotient U/s^2, where

$$U = \begin{vmatrix} \phi_s & \phi_{x_0'} & \phi_{y_0'} & \phi_{z_0'} \\ \psi_s & \psi_{x_0'} & \psi_{y_0'} & \psi_{z_0'} \\ \chi_s & \chi_{x_0'} & \chi_{y_0'} & \chi_{z_0'} \\ 0 & x_0' & y_0' & z_0' \end{vmatrix}.$$

This quotient is a finite and continuous function of its arguments even when $s=0$. For by applying the mean value theorem for the variable s to the elements of the minor of the element 0 in the determinant, and by using the relations (16), it is seen that U/s^2 approaches 1 as s approaches zero. If U/s^2 remains different from zero along an extremal arc C_{01} with the initial element $(x_0, y_0, z_0, x_0', y_0', z_0')$, it will also remain different from zero on C_{01} when $(x_0, y_0, z_0, x_0', y_0', z_0')$ are replaced by the values $(x_2, y_2, z_2, x_2', y_2', z_2')$ corresponding to a point 2 chosen on C_{01} sufficiently near to 0, and in the order 201 with the points 0 and 1.

The family of extremals through the initial point 0 of an extremal arc C_{01} can be found by making the constants x_0, y_0, z_0 in equations (14) equal to the coördinates of the point 0. If x_0', y_0', z_0' are replaced by functions $x'(u, v)$, $y'(u, v)$, $z'(u, v)$ which reduce to the coördinates of the initial direction of C_0 for $u = u_0$, $v = v_0$, and which satisfy equation (15) identically, the equations (14) take the form

$$(19) \qquad x = \phi(s, u, v), \qquad y = \psi(s, u, v), \qquad z = \chi(s, u, v).$$

These equations represent the extremal C_{01} for $u = u_0$, $v = v_0$, and satisfy also the identities

$$x_0 = \phi(0, u, v), \qquad y_0 = \psi(0, u, v), \qquad z_0 = \chi(0, u, v),$$

which express the fact that all of the extremals pass through the point (x_0, y_0, z_0) for $s = 0$. It is evident that the derivatives ϕ_u, ψ_u, χ_u and the similar ones for v all vanish identically for $s = 0$.

The functional determinant

$$\Delta(s, u, v) = \begin{vmatrix} \phi_s & \phi_u & \phi_v \\ \psi_s & \psi_u & \psi_v \\ \chi_s & \chi_u & \chi_v \end{vmatrix}$$

is readily seen to be equal to the product

$$U \begin{vmatrix} x' & x'_u & x'_v \\ y' & y'_u & y'_v \\ z' & z'_u & z'_v \end{vmatrix}.$$

The functions $x'(u, v), y'(u, v), z'(u, v)$ can always be chosen so that the second factor is different from zero for $u = u_0, v = v_0$, and under those circumstances $\Delta(s, u, v)$ vanishes for $s = 0$ on C_{01}, but is different from zero in the neighborhood of $s = 0$ on account of the properties of U derived above.

If the family of extremals (19) *through the point* 0 *includes the extremal* C_{01} *and has a determinant* Δ *which is different from zero everywhere on the arc* C_{10} *except at the point* 0, *then the quotient* U/s^2 *does not vanish on* C_{01}, *and a point* 2 *can be selected in the manner described in the first paragraph of this section, such that the family of extremals through* 2 *will have a determinant* Δ *which does not vanish anywhere on the arc* C_{01}.

An extremal to which a surface S is transversal can be imbedded in a two-parameter family of extremals each of which is cut transversally by the surface S. To show this it is only necessary to substitute for x_0, y_0, z_0 in the equations (14) of the extremals the functions $x(u, v), y(u, v), z(u, v)$ which define the surface, and for x'_0, y'_0, z'_0 the direction cosines of the directions to which the surface is transversal determined by equations (18). That such a direction is uniquely determined at each point of the surface can be seen as follows. The equations

$$(20)\quad x'^2+y'^2+z'^2=1, \qquad F_{x'}x_u+F_{y'}y_u+F_{z'}z_u=0, \qquad F_{x'}x_v+F_{y'}y_v+F_{z'}z_v=0,$$

are satisfied by one set of values corresponding to the intersection 0 of the extremal C_{01} with the surface S, and at the point 0 the functional determinant of the first members with respect to x', y', z' is different from zero. In fact this determinant is readily seen to be equal to the product of the two determinants

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & x' & y' & z' \\ 0 & x_u & y_u & z_u \\ 0 & x_v & y_v & z_v \end{vmatrix} \cdot \begin{vmatrix} 0 & x' & y' & z' \\ x' & F_{x'x'} & F_{x'y'} & F_{x'z'} \\ y' & F_{y'x'} & F_{y'y'} & F_{y'z'} \\ z' & F_{z'x'} & F_{z'y'} & F_{z'z'} \end{vmatrix}.$$

From equations (18)

$$(21)\quad y_u z_v - y_v z_u = kF_{x'}, \qquad z_u x_v - z_v x_u = kF_{y'}, \qquad x_u y_v - x_v y_u = kF_{z'},$$

so that on account of equation (4) the first determinant is equal to kF. The factor k is not zero if the surface S has not a singular point at 0, *and we add here the assumption* $F \neq 0$ *at the point* 0. The second determinant is different from zero on account of equation (7). From the well-known theorems on implicit functions it follows that x', y', z', are determined as functions of u and v, of class C' in the neighborhood of the point 0.

After substituting the values $x(u, v)$, $y(u, v)$, $z(u, v)$, $x'(u, v)$, $y'(u, v)$, $z'(u, v)$ as just determined for x_0, y_0, z_0, x_0', y_0', z_0', the equations (14) take the form

$$(22)\quad x = \phi(s, u, v), \qquad y = \psi(s, u, v), \qquad z = \chi(s, u, v).$$

On account of the properties of the functions (14) and of the solutions of equations (20) there will be a region

$$0 \leqq s \leqq s_1, \qquad |u - u_0| \leqq \kappa, \qquad |v - v_0| \leqq \kappa,$$

in which ϕ, ϕ_s, ψ, ψ_s, χ, χ_s are of class C', where the quantities u_0, v_0 are the particular parameter values defining the extremal C_{01}. Since for $s = 0$ the point (x, y, z) for all values of u and v lies on the surface S, the equations of S in the neighborhood of the point 0 are

$$x = \phi(0, u, v), \qquad y = \psi(0, u, v), \qquad z = \chi(0, u, v),$$

and on account of the transversality of S to the extremals of the family, we have

$$(23)\quad \begin{aligned} F_{x'}\phi_u + F_{y'}\psi_u + F_{z'}\chi_u &= 0, \\ F_{x'}\phi_v + F_{y'}\psi_v + F_{z'}\chi_v &= 0, \end{aligned}$$

for $s = 0$, where the arguments of the derivatives of F are $\phi, \psi, \chi, \phi_s, \psi_s, \chi_s$.

The value of the determinant

$$\Delta(s, u, v) = \begin{vmatrix} \phi_s & \phi_u & \phi_v \\ \psi_s & \psi_u & \psi_v \\ \chi_s & \chi_u & \chi_v \end{vmatrix}$$

when $s = 0$ becomes simply kF, with the help of the equations (4) and (23), and the factor k cannot vanish since it is understood that the surface S does not have a singular point at 0. And F has been assumed to be different from zero at the point 0 at least.

It has been shown therefore that *if a surface* S *is transversal to an extremal* C_{01} *at the point* 0, *then* C_{01} *can be imbedded in a family of extremals* (22) *to*

each of which S is transversal. The determinant $\Delta(s, u, v)$ *of the family does not vanish in the neighborhood of the point* 0.

Suppose that the determinant Δ vanishes for some point 2 different from O on the extremal C_{01}, corresponding to the values s_2, u_0, v_0. If it is assumed that at least one of the three-rowed determinants of the matrix

$$\begin{Vmatrix} \Delta_s & \Delta_u & \Delta_v \\ \phi_s & \phi_u & \phi_v \\ \psi_s & \psi_u & \psi_v \\ \chi_s & \chi_u & \chi_v \end{Vmatrix} \tag{24}$$

does not vanish with Δ, for convenience the determinant

$$\begin{vmatrix} \Delta_s & \Delta_u & \Delta_v \\ \phi_s & \phi_u & \phi_v \\ \psi_s & \psi_u & \psi_v \end{vmatrix},$$

then the equations

$$\Delta(s, u, v) = 0, \qquad \phi(s, u, v) = x, \qquad \psi(s, u, v) \quad y \tag{25}$$

determine s, u, v as functions

$$s = s(x, y), \qquad u = u(x, y), \qquad v = v(x, y) \tag{26}$$

in the neighborhood of the values x_2, y_2. By substituting these results in the third of the equations (22), a surface

$$z = z(x, y) \tag{D}$$

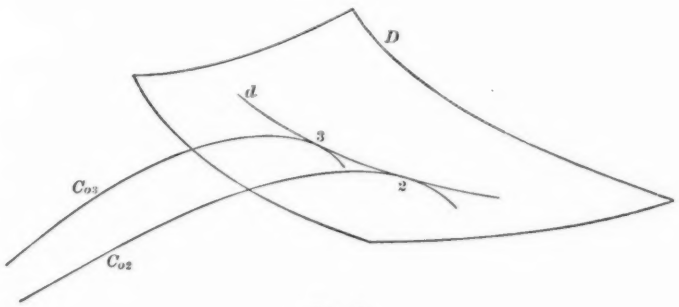

FIG. 1.

is found (See Figure 1). This surface is the envelope of the extremals (22). For from equations (25) and (D) the expression

$$\chi_s - z_x \phi_s - z_y \psi_s$$

has the factor

$$\begin{vmatrix} 0 & \Delta_s & \Delta_u & \Delta_v \\ \phi_s & \phi_s & \phi_u & \phi_v \\ \psi_s & \psi_s & \psi_u & \psi_v \\ \chi_s & \chi_s & \chi_u & \chi_v \end{vmatrix} = -\Delta\Delta_s,$$

which vanishes on D on account of the first of equations (25).

If a one-parameter family be chosen arbitrarily from the set (22), this family will not in general have an enveloping curve. The condition that a curve

$$(d) \qquad x = x(\alpha), \qquad y = y(\alpha), \qquad z = z[x(\alpha), y(\alpha)] = z(\alpha)$$

on the surface D shall be an envelope of extremals, may be derived as follows. If we substitute $x(\alpha)$, $y(\alpha)$ in the functions $u(x, y)$, $v(x, y)$ defined by equations (26), two functions $u(\alpha)$, $v(\alpha)$ are determined, and a one-parameter family of extremals is defined when $u(\alpha)$ and $v(\alpha)$ are substituted in (22). These extremals are tangent to the curve d if x and y are determined as functions of α so that

$$(27) \qquad x_\alpha = m\phi_s, \qquad y_\alpha = m\psi_s, \qquad z_\alpha = z_x x_\alpha + z_y y_\alpha = m\chi_s,$$

s, u, v being thought of as functions of x and y. The three determinants of the matrix

$$\begin{vmatrix} x_\alpha & y_\alpha & z_x x_\alpha + z_y y_\alpha \\ \phi_s & \psi_s & \chi_s \end{vmatrix}$$

must therefore be zero, i. e., the three equations

$$-\psi_s z_x x_\alpha + (\chi_s - \psi_s z_y) y_\alpha = 0,$$
$$(\phi_s z_x - \chi_s) x_\alpha + \phi_s z_y y_\alpha = 0,$$
$$\psi_s x_\alpha - \phi_s y_\alpha = 0$$

must be satisfied. The coefficients of x_α and y_α in these equations can not all vanish since at least one of the derivatives ϕ_s, ψ_s, χ_s is different from zero at the point 2. That any two of the equations are a consequence of the third may be shown by expanding the determinant of any pair of the equations, and using the relation

$$\chi_s - z_x\phi_s - z_y\psi_s = 0.$$

The determination of a one parameter family of extremals having an enveloping curve d is therefore to be effected by solving one of the above equations. It has the form

$$A(x, y)x_\alpha + B(x, y)y_\alpha = 0$$

when s, u, v are replaced by their values in terms of x and y from equations (26). Since this differential equation is of the first order there exists one and only one integral curve

$$x = x(\alpha), \qquad y = y(\alpha), \tag{28}$$

in the xy-plane, passing through the point (x_2, y_2) for $\alpha = 0$. The equations of the family of extremals tangent to d are found by substituting $x(\alpha)$, $y(\alpha)$ in the expressions for u and v in terms of x and y from equations (26), and then putting the resulting functions $u(\alpha)$, $v(\alpha)$ in equations (22). A family of extremals

$$x = \phi(s, \alpha), \qquad y = \psi(s, \alpha), \qquad z = \chi(s, \alpha) \tag{29}$$

is thus found, containing C_{01} for $\alpha = 0$, and tangent to d when $s = s(\alpha)$ from equations (28) and (26). The equations of the envelope d will then be

$$x = \phi[s(\alpha), \alpha], \qquad y = \psi[s(\alpha), \alpha], \qquad z = \chi[s(\alpha), \alpha]. \tag{30}$$

If necessary the sign of the parameter α may be changed so that the factor m in equations (27) is positive.

Consider then a family of extremals

$$x = \phi(s, u, v), \qquad y = \psi(s, u, v), \qquad z = \chi(s, u, v),$$

all of which pass through a fixed point, or which are all cut transversally by a given surface S, for $s = 0$. Suppose that on a particular extremal C_{02} the determinant Δ vanishes at the point 2 different from zero, but that one at least of the determinants of the matrix (24) is different from zero at 2. Then the family of extremals has an enveloping surface D (see Fig. 1), which touches C_{02} at 2 and for which 2 is not a singular point. On the surface D there exists a unique curve d without singular point, which passes through the point 2 and envelops a one-parameter family of extremals containing the extremal C_{02}.

§ 4. *Jacobi's necessary condition.*

By means of the one-parameter family of extremals and the enveloping curve d, found in the last section, it can be shown that an arc C_{01} which joins a fixed point 0 to another fixed point 1, or which joins 1 to a fixed surface S, and which furthermore minimizes the integral J, can not have upon it a contact point 2 with the enveloping surface D. It will be seen that the value of the integral J taken along an extremal C_{03} of the set (19) or (22) from the value $s = 0$ to the value $s(\alpha)$ at the contact point 3 of C_{03} with d, and then along d_{32} from the point 3 to the point 2, is independent of the particular extremal chosen, i. e., it is independent of the value of the parameter α defining the extremal. In fact, the derivative of $J(C_{03})$ is

$$\frac{dJ(C_{03})}{d\alpha} = \frac{d}{d\alpha}\int_{s_0}^{s_3} F(\phi, \psi, \chi, \phi_s, \psi_s, \chi_s)\,ds$$

$$= F\frac{ds}{d\alpha}\Big|^3 + \int_{s_0}^{s_3} [F_x\phi_\alpha + F_y\psi_\alpha + F_z\chi_\alpha + F_{x'}\phi_{\alpha s} + F_{y'}\psi_{\alpha s} + F_{z'}\chi_{\alpha s}]\,ds.$$

After integration by parts this takes the form

$$\frac{dJ(C_{03})}{d\alpha} = F\frac{ds}{d\alpha}\Big|^3 + [F_{x'}\phi_\alpha + F_{y'}\psi_\alpha + F_{z'}\chi_\alpha]_0^3,$$

the integral part vanishing on account of the fact that the curve C_{03} is an extremal. The terms in the bracket vanish for $s = 0$ when all the extremals (19) pass through a fixed point 0, for then

$$x_0 = \phi(0, \alpha), \qquad y_0 = \psi(0, \alpha), \qquad z_0 = \chi(0, \alpha),$$

and consequently ϕ_α, ψ_α, χ_α vanish for $s = 0$. They also cancel when the extremals are cut transversally by a surface S on account of equation (17), since in that case the equations

$$x = \phi(0, \alpha), \qquad y = \psi(0, \alpha), \qquad z = \chi(0, \alpha),$$

represent a curve on the surface S which is transversal to the extremals (29). On account of (4) the resulting equation may be written in the form

$$\frac{dJ(C_{03})}{d\alpha} = \left[\left(\phi_s\frac{ds}{d\alpha} + \phi_\alpha\right)F_{x'} + \left(\psi_s\frac{ds}{d\alpha} + \psi_\alpha\right)F_{y'} + \left(\chi_s\frac{ds}{d\alpha} + \chi_\alpha\right)F_{z'}\right]^3,$$

the arguments of the derivatives of F being $\phi, \psi, \chi, \phi_s, \psi_s, \chi_s$. From equations (30), therefore,

$$\frac{dJ(C_{03})}{d\alpha} = F_{x'}x_\alpha + F_{y'}y_\alpha + F_{z'}z_\alpha|^3, \tag{31}$$

where x_α, y_α, z_α determine the direction of the curve d at the point 3. Since the positive tangent to d coincides with the positive tangent to C_{03}, we have

$$\frac{dJ(C_{03})}{d\alpha} = F(x, y, z, x_\alpha, y_\alpha, z_\alpha)|^3. \tag{32}$$

This follows because from equations (3) and (27) the coefficient $F_{x'}$ in equation (31) becomes

$$F_{x'}(\phi, \psi, \chi, \phi_s, \psi_s, \chi_s)|^3 = F_{x'}(x, y, z, x_\alpha, y_\alpha, z_\alpha)|^3.$$

The derivative of the integral $J(d_{32})$ is

$$\frac{dJ(d_{32})}{d\alpha} = \frac{d}{d\alpha}\int_{\alpha_3}^{\alpha_2} F(x, y, z, x_\alpha, y_\alpha, z_\alpha)\,d\alpha = -F(x, y, z, x_\alpha, y_\alpha, z_\alpha)|^3 \tag{33}$$

From equations (32) and (33) therefore,

$$\frac{d}{d\alpha}[J(C_{03}) + J(d_{32})] = 0,$$

from which the following generalization of KNESER's theorem in the plane is obtained:

Suppose that an extremal C_{01} is contained in a one-parameter family of extremals, each of which passes through a fixed point 0 and which have an enveloping curve touching C_{01} at a point 2 (see Figure 1). Then if 3 is a neighboring point to 2 on d, the value of J taken along C_{03} plus the value of J taken along d_{32} is always equal to the value of J taken along C_{02}. In other words

$$J(C_{03}) + J(d_{32}) = J(C_{02}).$$

When the extremals are all cut transversally by a surface S, the same equation is true if 0 is understood to denote the variable intersection of C_{03} with S.

The envelope d_{32} can not satisfy Euler's equations, for it was shown in § 1 that but one solution of these equations passes through a given point in a given direction, and d_{32} would have to coincide, therefore, with the extremal C_{02} in the neighborhood of the point 2. This contradicts the hypothesis that the point 2 is the nearest point to 0 on the arc C_{01}, for which Δ vanishes. It follows that the points 3 and 2 can always be joined by a curve d'_{32} such that

$$J(C_{03}) + J(d'_{32}) < J(C_{02}).$$

*If therefore an extremal C_{01} minimizes or maximizes the integral J, it must not have upon it a point 2 which is conjugate to the initial point 0.**

§ 5. *The necessary conditions of Weierstrass and Legendre.*

It has been shown in the preceding sections that if a curve C_{01} joins two fixed points, or a fixed surface with a fixed point, and minimizes the integral J, it must be a solution of Euler's equations and must satisfy Jacobi's necessary condition. A further condition similar to the one discovered by LEGENDRE for a weak minimum, and extended by WEIERSTRASS for the case of a strong minimum, must also be satisfied.

Let us consider an extremal arc C_{01},

$$x = x(t), \qquad y = y(t), \qquad z = z(t),$$

and a curve V

$$x = x(u), \qquad y = y(u), \qquad z = z(u),$$

* The theorem has been proved for conjugate points 2 at which not all of the determinants of the matrix (24) vanish with Δ. By using the so-called second variation it can be shown that even without this restriction the point 2 must not lie between 0 and 1 on C_{01}. The case where the point 2 coincides with 1 and all determinants of the matrix (24) vanish is an exceptional one, which has been discussed for problems in the plane but not for problems in space.

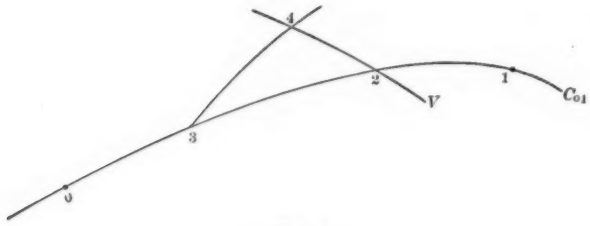

FIG. 2.

intersecting C_{01} at a point 2. Through a point 3 on C_{01} near to 2, there can always be found a family of curves

$$x = f(t, u), \qquad y = g(t, u), \qquad z = h(t, u), \tag{34}$$

containing C_{32} for $u = u_2$, which furthermore pass through 3 for the parameter value $t = t_3$ and intersect V for $t = t_2$. Analytically, then,

$$\begin{aligned} &f(t_2, u) \equiv x(u), \qquad g(t_2, u) \equiv y(u), \qquad h(t_2, u) \equiv z(u), \\ &x_3 \equiv f(t_3, u), \qquad y_3 \equiv g(t_3, u), \qquad z_3 \equiv h(t_3, u). \end{aligned} \tag{35}$$

The sum $J_{34} + J_{42}$ is now a function of the parameter u, whose derivative for $u = u_2$ can be readily calculated. The derivative of J_{34} is

$$\frac{dJ_{34}}{du} = \int_{t_3}^{t_2} \{F_x f_u + F_y g_u + F_z h_u + F_{x'} f_{tu} + F_{y'} g_{tu} + F_{z'} h_{tu}\} dt,$$

where the arguments of F and its derivatives are f, g, h, f_t, g_t, h_t. For $u = u_2$, this becomes, by the usual integration by parts of the calculus of variations and with the help of the identities (35),

$$\left[\frac{dJ_{34}}{du}\right]_{u=u_2} = F_{x'} x_u + F_{y'} y_u + F_{z'} z_u \Big|^2,$$

since C_{32} is an extremal. Here the arguments x, y, z, x', y', z' of the derivatives of F refer to C_{32} since the curves (34) go over into C_{32} for $u = u_2$, and x_u, y_u, z_u define the direction of V. The derivative of J_{42} is simply

$$\left[\frac{dJ_{42}}{du}\right]_{u=u_2} = -F(x, y, z, x_u, y_u, z_u) \Big|^2.$$

The derivative of the sum $J_{34} + J_{42}$ is therefore

$$\left[\frac{d}{du}(J_{34} + J_{42})\right]_{u=u_0} = -E(x, y, z, x', y', z', x_u, y_u, z_u) \Big|^2, \tag{36}$$

where

$$\begin{aligned} E = F(x, y, z, x_u, y_u, z_u) &- x_u F_{x'}(x, y, z, x', y', z') \\ &- y_u F_{y'}(x, y, z, x', y', z') - z_u F_{z'}(x, y, z, x', y', z'). \end{aligned} \tag{37}$$

From equation (36) the following theorem similar to that of WEIERSTRASS for the plane case, can be derived.

If the arc C_{01} furnishes J a strong minimum, the function E must be $\geqq 0$ for any set of values x, y, z, x', y', z' on C_{01} and for all directions x_u, y_u, z_u; while if C_{01} furnishes only a weak minimum, the E-function must similarly be $\geqq 0$ at least for directions x_u, y_u, z_u in some neighborhood of the directions x', y', z' on C_{01}. (For a maximum E must be $\leqq 0$.)

The theorem is evidently true, for if the E-function were negative at a point 2 for some direction x_u, y_u, z_u, then on account of equation (36) the inequality

$$J_{34} + J_{42} < J_{32}$$

would hold for values of u sufficiently near to u_2.

On account of the homogeneity of the function F, the expression (37) may be written in the form

$$\begin{aligned} E(x, y, z, x', y', z', x_u, y_u, z_u) &= F(x, y, z, x_u, y_u, z_u) - F \\ &\quad - (x_u - x')F_{x'} - (y_u - y')F_{y'} - (z_u - z')F_{z'} \\ &= \tfrac{1}{2}\{(x_u - x')^2 F_{x'x'} + (y_u - y')^2 F_{y'y'} + (z_u - z')^2 F_{z'z'} \\ &\quad + 2(y_u - y')(z_u - z')F_{y'z'} + 2(z_u - z')(x_u - x')F_{z'x'} + 2(x_u - x')(y_u - y')F_{x'y'}\}, \end{aligned} \tag{38}$$

the arguments of the second derivatives of F being

$$x, y, z, x' + \theta(x_u - x'), y' + \theta(y_u - y'), z' + \theta(z_u - z'), (0 < \theta < 1).$$

We can derive therefore the following Legendre condition:

If the extremal arc C_{01} furnishes either a strong or a weak minimum, the quadratic form

$$Q = F_{x'x'}\xi^2 + F_{y'y'}\eta^2 + F_{z'z'}\zeta^2 + 2\eta\zeta F_{y'z'} + 2\zeta\xi F_{z'x'} + 2\xi\eta F_{x'y'}, \tag{39}$$

must be $\geqq 0$ for all values of x, y, z, x', y', z' on the arc C_{01}, and for all values of ξ, η, ζ.

For if ξ, η, ζ were a set of values making Q negative at some point of C_{01}, the values x_u, y_u, z_u could be taken so that

$$x_u - x' = \epsilon\xi, \; y_u - y' = \epsilon\eta, \; z_u - z' = \epsilon\zeta,$$

and for a sufficiently small ϵ the expression (38) would also be negative.

These necessary conditions may also be stated in terms of the functions F_1, $F_{x'x'}$, $F_{y'y'}$, $F_{z'z'}$. From (39) and the equations (6) it follows that F_1 can not be negative. For in that case the binary form obtained from Q by putting $\xi = 0$, for example, would be an indefinite form. The derivatives $F_{x'x'}$, $F_{y'y'}$, $F_{z'z'}$, if different from zero, must have the same sign as that of Q, as is easily

seen by putting $\eta = \zeta = 0$ in (39). Conversely if $F_1 > 0$, and $F_{x'x'}$, $F_{y'y'}$, $F_{z'z'}$ are not all zero, the identities

$$
\begin{aligned}
F_{x'x'}Q &= [F_{x'x'}\xi + F_{x'y'}\eta + F_{x'z'}\zeta]^2 + (y'\zeta - z'\eta)^2 F_1, \\
F_{y'y'}Q &= [F_{y'x'}\xi + F_{y'y'}\eta + F_{y'z'}\zeta]^2 + (z'\xi - x'\zeta)^2 F_1, \\
F_{z'z'}Q &= [F_{z'x'}\xi + F_{z'y'}\eta + F_{z'z'}\zeta]^2 + (x'\eta - y'\xi)^2 F_1,
\end{aligned} \tag{40}
$$

which are easily verified, show that such of the quantities Q, $F_{x'x'}$, $F_{y'y'}$, $F_{z'z'}$ as are different from zero have the same sign at all points of the extremal C_{01}, whatever be the values of ξ, η, ζ.

It should be noticed that when $F_1 > 0$ the derivative $F_{x'x'}$ vanishes when and only when $y' = z' = 0$, that is, in the direction of the x-axis. That it is zero for these arguments is seen from (5) since $x' \neq 0$. But from (6) it follows that y' and z' must vanish with $F_{x'x'}$ since $F_1 > 0$. Furthermore $F_{x'x'}$ cannot assume opposite signs, since it is possible to pass continuously from one set of values (x, y, z, x', y', z') to any other without passing through values for which $y' = z' = 0$. The Legendre necessary condition may be restated, then, in the following form:

If for a regular problem ($F_1 \neq 0$) *the extremal* C_{01} *furnishes either a maximum or a minimum for the integral* J, F_1 *must be positive. Furthermore the functions* $F_{x'x'}$, $F_{y'y'}$, $F_{z'z'}$, *which vanish only in the direction of the* x, y, *and* z*-axes, respectively, must be positive for a minimum and negative for a maximum.*

§ 6. *The invariant integral. Sufficient conditions.*

In § 3 it was shown that an extremal arc which does not contain a conjugate point 2 to 0, either between 0 and 1 or at 1, can be imbedded in a two-parameter family of extremals through a point 3 near to 0, whose determinant Δ does not vanish anywhere on C_{01}. Consider either this family or the family (22) to which the surface S is transversal. Suppose that the extremal C corresponds to the values $u = u_0$, $v = v_0$, and that the determinant $\Delta(s, u_0, v_0)$ does not vanish for any point of the arc C_{01}, i. e., that the enveloping surface D does not touch the arc C_{01}. Then the set of extremals forms a field about the extremal C. A positive constant δ may be chosen so small that the extremals of the set for which $|u - u_0| \leqq \delta$, $|v - v_0| \leqq \delta$ sweep out a portion of space which contains C in its interior, and through each point of which there passes one and only one extremal of the set. The constant δ can be so restricted that in this field the determinant Δ does not vanish, and the equations

$$
\phi(s, u, v) = x, \qquad \psi(s, u, v) = y, \qquad \chi(s, u, v) = z \tag{41}
$$

have unique solutions

$$(42) \qquad s = s(x, y, z), \qquad u = u(x, y, z), \qquad v = v(x, y, z)$$

of class C'.*

Consider a curve K

$$(43) \qquad x = \bar{x}(t), \qquad y = \bar{y}(t), \qquad z = \bar{z}(t)$$

of class D', which lies entirely in the field. By means of equations (42) and (43) u and v are determined as functions of t. If these functions be substituted in the equations (41) there results a one-parameter family of extremals,

$$(44) \qquad x = \Phi(s, t), \qquad y = \Psi(s, t), \qquad z = \mathrm{X}(s, t),$$

having as parameter of the family the parameter t of the curve K. For $s = 0$ these extremals all pass through the point 3 or else intersect the surface S, which is transversal to all the extremals of the set (41) along a curve k.

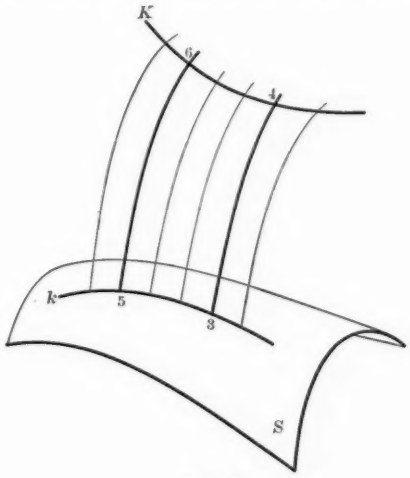

FIG. 3.

The value of the integral J, carried along an extremal of the set (44) from the point 3 where $s = 0$ to its point of intersection 4 with K, is a function of t, the parameter of the set. Using the same reasoning as that of the preceding section, the derivative of this function is found to be

$$\frac{dJ_{34}}{dt} = \frac{d}{dt}\int_0^{s_4} F(\Phi, \Psi, \mathrm{X}, \Phi', \Psi', \mathrm{X}')\,ds = \bar{x}'F_{x'} + \bar{y}'F_{y'} + \bar{z}'F_{z'}|^4.$$

The value of the second member vanishes at the lower limit 3, since all the

* The proof of these statements may be made by a method analogous to that given by BOLZA for the case of the plane: *Lectures on the Calculus of Variations*, Chicago, 1904, § 34.

extremals of the set pass through a fixed point 3 or else are cut transversally by the curve k. The arguments of the derivatives of F are the functions $\Phi, \Psi, X, \Phi', \Psi', X'$ at the point 4 on K, or what is the same thing, the values of $\phi, \psi, \chi, \phi', \psi', \chi'$ with s, u, v expressed as functions of t from equations (42) and (43). Let t_4 be the value of t which gives the extremal connecting 3 with 4, and t_6 the value giving the extremal which connects a point 5 of k with the point 6 of K, and let the above equation be integrated from $t = t_4$ to $t = t_6$. Then

$$J_{56} - J_{34} = \int_{t_4}^{t_6} \{\bar{x}' F_{x'} + \bar{y}' F_{y'} + \bar{z}' F_{z'}\}\, dt.$$

Now the first member of this equation is entirely independent of the form of the curve K. It follows that this is true of the second member also, in other words: *The value of the integral*

$$J^* = \int \{\bar{x}' F_{x'} + \bar{y} F_{y'} + \bar{z}' F_{z'}\}\, dt,$$

taken along a curve K in the field, is independent of the path K, and depends only on the endpoints. The derivatives of F are here thought of as functions of x, y, z by means of (42). The integral has the further properties *that $J^* = 0$ if the curve K is at each point transversal to the extremal of the field, and that J^* reduces to J if K is an extremal.* These properties follow easily from equations (17) and (4).

The integral J^* is the generalization of Hilbert's invariant integral. It is useful in comparing the value $J(C_{01})$ taken along the extremal C with the value of J taken along any other curve $\bar{C}$

$$x = \bar{x}(t), \qquad y = \bar{y}(t), \qquad z = \bar{z}(t)$$

which joins the fixed point 0, or a point 0′ of the surface S, with the fixed point 1. For the case of the transversal surface S, let $k_{00'}$ be any curve which lies entirely on S and in the field, and which joins 0 with 0′. This curve is transversal to the extremals of the field. Then from the properties of the integral J^*,

$$\begin{aligned} J(\bar{C}_{0'1}) - J(C_{01}) &= J(\bar{C}_{0'1}) - J^*(C_{01}) \\ &= J(\bar{C}_{0'1}) - J^*(k_{00'}) - J^*(\bar{C}_{0'1}) \\ &= J(\bar{C}_{0'1}) - J^*(\bar{C}_{0'1}). \end{aligned}$$

Hence by making use of the E-function defined in equation (37), it follows that

$$(45) \qquad J(\bar{C}_{0'1}) - J(C_{01}) = \int_{\bar{C}_{0'1}} E(x, y, z, \phi', \psi', \chi', \bar{x}', \bar{y}', \bar{z}')\, dt,$$

where ϕ', ψ', χ' are thought of as functions of x, y, z as explained above. Evidently if $E \geqq 0$ along the curve $\bar{C}$, then

$$J(\bar{C}_{0'1}) \geqq J(C_{01}).$$

The same conclusion holds when C passes through two fixed points 0 and 1.

By a method similar to that used in § 5 applied to the expression (38), the E-function can be expressed in any one of three forms similar to those in (40). The first of these is

$$2F_{x'x'} \cdot E = [(x_u - x')F_{x'x'} + (y_u - y')F_{x'y'} + (z_u - z')F_{x'z'}]^2 + (y'z_u - z'y_u)^2 F_1.$$

Now the Legendre condition of § 5 is supposed to be satisfied, i. e., $F_1 > 0$, while $F_{x'x'}$, $F_{y'y'}$, $F_{z'z'}$ are not negative and vanish only in the coördinate directions respectively.

It follows for a regular problem, in which Legendre's necessary condition for a minimum is satisfied, that the Weierstrass E-function (37) *is never negative, and is equal to zero only when* $x_u : y_u : z_u = x' : y' : z'$.*

Hence in the field of extremals under consideration the E-function vanishes only when the direction of the comparison curve $\bar{C}$ at the point x, y, z coincides with the direction of the extremal of the field which passes through that point. Therefore from (45)

$$J(\bar{C}_{0'1}) > J(C_{01}),$$

unless the direction of $\bar{C}_{0'1}$ coincides at every point with the direction of the extremal of the field through that point. But this can occur only when $\bar{C}_{0'1}$ coincides throughout its entire extent with the extremal C_{01}. For along the curve $\bar{C}_{0'1}$ the parameters u and v are functions of t whose derivatives, from equations (42), are

$$\frac{du}{dt} = u_x \bar{x}' + u_y \bar{y}' + u_z \bar{z}',$$

$$\frac{dv}{dt} = v_x \bar{x}' + v_y \bar{y}' + v_z \bar{z}'.$$

Since u and v are constant along any one of the extremals of the field, the values of these derivatives are zero when the direction $(\bar{x}', \bar{y}', \bar{z}')$ coincides with that of one of the extremals. Hence if the direction of $C_{0'1}$ always coincides with the direction of some extremal, the second members of the above equations vanish identically, and along $\bar{C}_{0'1}$ the values of u and v must be constant. In other words, $\bar{C}_{0'1}$ must be an extremal, and since it passes through the point 1 it must be the extremal C_{01} itself. These results may be summarized in the following theorem.

* The expression (37) shows that E vanishes when the directions $x_u : y_u : z_u$ and $x' : y' : z'$ have the same sense, but not necessarily when they are opposite.

An extremal arc C_{01} joins two fixed points 0 and 1, or joins 1 to a surface S which cuts C_{01} transversally. For a regular problem ($F_1 \neq 0$) the integral J will be minimized by this arc in either case provided that Legendre's necessary condition for a minimum is satisfied along C_{01}, and that C_{01} has upon it no point conjugate to the fixed end-point 0 in the former case, or to the surface S in the latter.

§ 7. *Necessary conditions when one end point is variable on a curve.*

The results of the preceding sections apply with some modifications to the case where the end point 0 may vary on a given curve L,

$$(L) \qquad x = x(u), \qquad y = y(u), \qquad z = z(u),$$

instead of on a surface.

The curve C_{01} which minimizes the integral J with respect to other curves joining the curve L with the fixed point 1 must in the first place be an extremal which is cut by L transversally, i. e., at the point of intersection 0 of C_{01} with L, the equation

$$l_0 F_{x'} + m_0 F_{y'} + n_0 F_{z'} = 0$$

must be satisfied, where l_0, m_0, n_0 are the values of the direction cosines l, m, n of L at the point 0, and the arguments of $F_{x'}$, $F_{y'}$, $F_{z'}$ are the values of x, y, z, x', y', z' on the extremal C_{01} at this point.

The extremal C_{01}, cut transversally by L at 0, may be imbedded in a two-parameter family of extremals, all cut transversally by L. The equations

$$(46) \qquad \begin{aligned} x'^2 + y'^2 + z'^2 &= 1, \\ lF_{x'} + mF_{y'} + nF_{z'} &= 0, \end{aligned}$$

have by hypothesis the solution x'_0, y'_0, z'_0 which is given by the values of x', y', z' on C_{01} at the point 0. Let (α, β, γ), (λ, μ, ν) denote the direction cosines of the principal normal and binormal respectively along L. Let v be a variable such that

$$\alpha \cos v + \lambda \sin v = a(u, v), \quad \beta \cos v + \mu \sin v = b(u, v), \quad \gamma \cos v + \nu \sin v = c(u, v)$$

are the direction cosines of the perpendicular to (x', y', z') and to (l, m, n). Then

$$(47) \qquad x'a + y'b + z'c = 0.$$

The three equations (46) and (47) can now be solved for x', y', z' as functions of v and of the parameter u of the curve L, in the neighborhood of the values u_0, v_0 which correspond to the point 0 and the direction of the extremal C_{01}. The functional determinant of the left members of the equations with respect to x', y', z' is

$$\begin{vmatrix} 2x' & 2y' & 2z' \\ lF_{x'x'} + mF_{x'y'} + nF_{x'z'} & lF_{y'x'} + mF_{y'y'} + nF_{y'z'} & lF_{z'x'} + mF_{z'y'} + nF_{z'z'} \\ a & b & c \end{vmatrix}.$$

Let θ be the angle between (x', y', z') and (l, m, n). Since these directions are perpendicular to the direction (a, b, c) it follows that

$$(48) \quad ny' - mz' = a \sin\theta, \qquad lz' - nx' = b \sin\theta, \qquad mx' - ly' = c \sin\theta.$$

The value of $\sin\theta$ cannot be zero, for, if it were, F would vanish on account of the second equation of (46) and (4), and F has been assumed different from

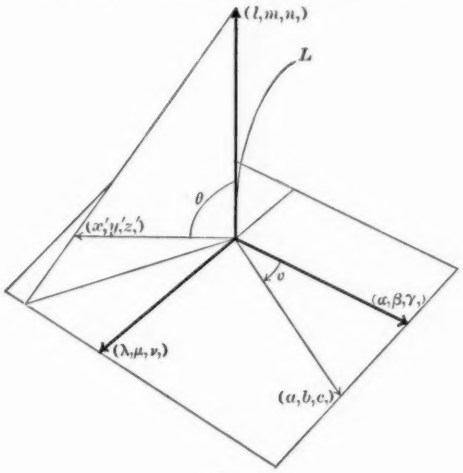

FIG. 4.

zero. The value of the above functional determinant, when calculated by the aid of (48) is twice the quadratic form

$$Q(l, m, n) = l^2F_{x'x'} + m^2F_{y'y'} + n^2F_{z'z'} + 2mnF_{y'z'} + 2nlF_{z'x'} + 2lmF_{x'y'}$$

divided by $\sin\theta$. This form vanishes only when the directions (l, m, n) and (x', y', z') coincide, or are opposite, i. e., only when $\sin\theta = 0$, and this cannot happen, as was seen above.

If the values $x(u)$, $y(u)$, $z(u)$ along the curve L, and the functions $x'(u, v)$, $y'(u, v)$, $z'(u, v)$ determined as above from equations (46), (47) *are substituted for x_0, y_0, z_0, x_0', y_0', z_0' in the equations of the extremals* (14), *a family of extremals*

$$x = \phi(s, u, v), \qquad y = \psi(s, u, v), \qquad z = \chi(s, u, v)$$

is determined, each of which is cut transversally by the curve L. The extremal C_{01} is represented for the particular parameter values u_0, v_0.

The value of the determinant

$$\Delta = \begin{vmatrix} \phi_s & \phi_u & \phi_v \\ \psi_s & \psi_u & \psi_v \\ \chi_s & \chi_u & \chi_v \end{vmatrix}$$

is in this case equal to zero at the point of intersection of L and C_{01}, since for $s = 0$ the functions ϕ, ψ, χ are identically equal to $x(u)$, $y(u)$, $z(u)$, and the derivatives ϕ_v, ψ_v, χ_v all vanish identically. But the value $s = 0$ is an isolated zero of Δ, whatever the values of u and v, for it can be shown that the derivative Δ_s does not vanish when $s = 0$. In fact the value of Δ_s for $s = 0$ is

$$\Delta_s = \begin{vmatrix} \phi_s & \phi_u & \phi_{sv} \\ \psi_s & \psi_u & \psi_{sv} \\ \chi_s & \chi_u & \chi_{sv} \end{vmatrix} = \begin{vmatrix} x'(u,v) & l(u) & x'_v(u,v) \\ y'(u,v) & m(u) & y'_v(u,v) \\ z'(u,v) & n(u) & z'_v(u,v) \end{vmatrix} \sqrt{\phi_u^2 + \psi_u^2 + \chi_u^2},$$

or, on account of (48),

$$\Delta_s = \sin\theta \sqrt{\phi_u^2 + \psi_u^2 + \chi_u^2}(x'_v a + y'_v b + z'_v c).$$

Now equation (47) holds identically in v, so that on differentiating (47) with respect to v the third factor in the above equation is seen from the expressions for $a(u, v)$, $b(u, v)$, $c(u, v)$ to be equal to

$$-(x'a_v + y'b_v + z'c_v) = -\left[x'a\left(u, v + \frac{\pi}{2}\right) + y'b\left(u, v + \frac{\pi}{2}\right) + z'c\left(u, v + \frac{\pi}{2}\right)\right],$$

and from the figure it follows that this is equal to $-\sin\theta$. The value of Δ_s for $s = 0$ is therefore

$$\Delta_s = -\sin^2\theta \sqrt{\phi_u^2 + \psi_u^2 + \chi_u^2}.$$

Since (x', y', z') can never coincide with (l, m, n), as was seen before, it follows that Δ_s *is different from zero for* $s = 0$, *and consequently the value* $s = 0$ *is an isolated zero of* $\Delta(s, u, v)$.

Consider now the extremal C_{01} of the family, corresponding to $u = u_0$, $v = v_0$, which intersects L at 0 and passes through the point 1. There will be an arc of C_{01} in the neighborhood of the point 0 along which Δ does not vanish. If Δ vanishes while one at least of the determinants of the matrix (24) is different from zero, at some point other than 0 on C_{01}, then the equation

$$\Delta(s, u, v) = 0$$

defines an enveloping surface D for the extremals of the family. Exactly as in § 3 a one-parameter family can be selected which has an enveloping curve d on

D, and an arc joining L to the point 1 can be found which gives to the integral J a smaller value than is given by C_{01}.

The following are therefore necessary conditions for a minimum with respect to curves joining a fixed curve L with a fixed point 1 *when the problem at hand is a regular one. The minimizing arc must be an extremal C_{01} which is cut transversally by the curve L at the intersection point* 0 *of L with C_{01}. The function F_1 must be positive, and the functions $F_{x'x'}$, $F_{y'y'}$, $F_{z'z'}$ must not be negative along C_{01}. Finally the envelope of the family of extremals to which L is transversal must not touch the arc C_{01}.**

§ 8. *Determination of a field of extremals to which a curve is transversal.*

The construction of a field for the case of an endpoint variable on a curve is somewhat more complicated than for the case of an endpoint variable on a surface. It has been shown that an extremal C_{01} to which L is transversal can be imbedded in a two-parameter family of neighboring extremals all cut transversally by L. These extremals were obtained by applying theorems on implicit functions to equations (46), (47), and consequently not only cut L at points in the neighborhood of the point 0, but have initial directions at their intersections with L which must be near to the direction of C_{01} at the point 0. In order to construct a field entirely surrounding L in the neighborhood of the point 0 it will be shown that the family can be extended to include extremals whose directions are not thus restricted, that is, that equations (46), (47) have unique solutions for all values of u near u_0 and all values of v between 0 and 2π, and furthermore that these solutions are periodic in v with period 2π.

In the first place, in any plane through a point x, y, z and containing the direction (l, m, n) there are at most two directions (x', y', z') to which (l, m, n) is transversal. For let (x'_0, y'_0, z'_0) be one such direction. Then any other direction in the plane has the direction cosines

$$x'_0 \cos\theta + l \sin\theta, \qquad y'_0 \cos\theta + m \sin\theta, \qquad z'_0 \cos\theta + n \sin\theta.$$

The expression

$$\Phi = lF_{x'} + mF_{y'} + nF_{z'}$$

with these arguments in $F_{x'}$, $F_{y'}$, $F_{z'}$ has the derivative

$$\frac{d\Phi}{d\theta} = l(x'_\theta F_{x'x'} + y'_\theta F_{x'y'} + z'_\theta F_{x'z'}) + m(x'_\theta F_{y'x'} + y'_\theta F_{y'y'} + z'_\theta F_{y'z'}) + n(x'_\theta F_{z'x'} + y'_\theta F_{z'y'} + z'_\theta F_{z'z'}).$$

This derivative vanishes when $\theta = \pi/2$ or $3\pi/2$ on account of equations (5).

* The exceptional case where the envelope has a singular point needs, as before, further consideration.

For other values of θ the expression takes the following form, obtained by substituting the values of x'_θ, y'_θ, z'_θ and using equations (5):

$$\frac{d\Phi}{d\theta} = \sec\theta(l^2F_{x'x'} + m^2F_{y'y'} + n^2F_{z'z'} + 2mnF_{y'z'} + 2nlF_{z'x'} + 2lmF_{x'y'}).$$

On account of the properties of this quadratic form, as shown in § 5, it is seen that the function $\Phi(\theta)$ is zero for $\theta = 0, 2\pi$ and has a derivative which changes sign only when $\theta = \pi/2, 3\pi/2$. Consequently, since Φ is periodic, it vanishes once between $\theta = \pi/2$ and $\theta = 3\pi/2$ and has no other zeros except $\theta = 0, 2\pi$. Hence in the plane through the point (x, y, z) and containing the directions (l, m, n) and (x'_0, y'_0, z'_0) there is but one other direction besides (x'_0, y'_0, z'_0) to which (l, m, n) is transversal, and that direction lies on the opposite side of the line (l, m, n) from (x'_0, y'_0, z'_0).

With these results in mind it is evident from the theory of implicit functions that the solutions x', y', z' of the equations (46) and (47), for a constant value of u, can be extended step by step over the whole interval from v_0 to $v_0 + 2\pi$. For if this were not the case, and an upper bound $V < v_0 + 2\pi$ for the extension existed, then the solutions x', y', z' would have at least one set of limiting values X', Y', Z' as v approached V. But any set of limiting values must be a solution of the equations, and since there can be but one solution in the half plane corresponding to $v = V$, it follows that X', Y', Z' form the only set. The solutions could therefore be extended beyond the value $v = V$, which is contrary to assumption concerning V. It is furthermore evident that the values of x', y', z' for $v = v_0 + 2\pi$ coincide with the values x'_0, y'_0, z'_0. These results may be stated in the following theorem:

*If a fixed line through the point (x, y, z) in the direction (l, m, n) is transversal to a single line (x'_0, y'_0, z'_0) through this point, then each half plane through the line (l, m, n) contains one and only one line (x', y', z') to which (l, m, n) is transversal at (x, y, z). The direction cosines of these lines are given by functions $x'(v)$, $y'(v)$, $z'(v)$ of class C', where v is the angle which determines the position of the half plane.**

Since for any particular value of u in the neighborhood of u_0 the equations (46), (47) have a unique solution for each value of v in the interval $(v_0, v_0+2\pi)$, it follows immediately from the theory of implicit functions that the totality of solutions $x'(u, v)$, $y'(u, v)$, $z'(u, v)$ so defined are functions of u and v of class C'.

* It follows readily that *any surface element through a point (x, y, z) is transversal to at most one direction on each side of the element.* Suppose there were two directions on the same side of a surface element E to which E is transversal at the point P. Then every line in E through P would be transversal to the two directions, in particular the line L common to E and to the plane determined by the two directions. But this would contradict the theorem proved above, since the two directions would lie in the same half plane through L.

There exists, therefore, a two-parameter family of extremals

$$x = \phi(s, u, v), \qquad y = \psi(s, u, v), \qquad z = \chi(s, u, v),$$

each cut transversally for $s = 0$ *by the curve* L, *where the functions* ϕ, ψ, χ *are defined in a region*

$$0 \leqq s \leqq S, \qquad |u - u_0| \leqq \delta, \qquad 0 \leqq v \leqq 2\pi,$$

and are periodic in v *with the period* 2π.

It is readily seen from the properties of the solutions of the Euler equations that the function ϕ, and similarly for ψ and χ, is continuous together with all its first derivatives and all its second derivatives that are formed by differentiating ϕ_s with respect to s, u, or v.

§ 9. *Sufficient conditions when one end point is variable on a curve.*

Consider a region in space containing in its interior the arc C_{01} to which the curve L is transversal at the point 0. This region may be restricted in the first place, so that through any point of it not on L there passes one and but one of the extremals determined in the preceding section, to which L is transversal at points in the neighboring of the point 0; and, in the second place, so that the determinant Δ is different from zero for values of s, u, v corresponding to points in the field.* The equations

$$\phi(s, u, v) = x, \qquad \psi(s, u, v) = y, \qquad \chi(s, u, v) = z \tag{49}$$

have a unique solution

$$s = s(x, y, z), \qquad u = u(x, y, z), \qquad v = v(x, y, z) \tag{50}$$

of class C', for all values of x, y, z in the field except those which define a point on L. The reasoning of § 6 now shows that the value of the integral

$$J^* = \int (\bar{x}' F_{x'} + \bar{y}' F_{y'} + \bar{z}' F_{z'}) dt$$

taken along a curve K of class D'

$$x = \bar{x}(t), \qquad y = y(t), \qquad z = \bar{z}(t) \tag{51}$$

which lies entirely in the field and has no point in common with L, depends only on the end points, and not at all on the form of the curve K. The derivatives of F are considered as functions of x, y, and z by means of equations (49), (50), and of t by (51). The integrand of J^*, thought of as a function of $x, y, z, \bar{x}', \bar{y}', \bar{z}'$, is not defined for points on the curve L since the function v is itself not defined for these points. The integral J^* has however a defi-

* The proof of the existence of such a field has been made by the writers and will be submitted to the Transactions for publication in the near future.

nite value when taken along a curve which has a finite number of points in common with L, since the integrand is limited and the only possible discontinuities are at the intersections with L. Let the value of J^* be defined as zero when K coincides with a portion of L. Then it may readily be seen that the integral is independent of the path for all curves which lie in the field, including those which intersect L or have an arc in common with L.

For let K and K_1 be two curves joining the same two points, where K_1 has no point in common with L. Suppose in the first place that K intersects L at a point P but has no other point in common with L. Let $\bar{K}$ be a curve which has no point in common with L and which coincides with K except along an arc which includes the point P. Then $J^*(\bar{K}) = J^*(K_1)$, and since by taking $\bar{K}$ near enough to K the value of $J^*(\bar{K})$ may be made to differ from $J^*(K)$ by an arbitrarily small amount, it follows also that $J^*(K) = J^*(K_1)$. Suppose now that K has an arc K_{56} which coincides with L. Then by definition $J^*(K_{56}) = 0$. Let $\bar{K}_{56}$ be a curve in the field which joins 5 to 6 but has no other points in common with L. Recalling the meaning of J^* as derived in § 5, it is seen that $J^*(\bar{K}_{56}) = 0$. In fact $J^*(\bar{K}_{56})$ is equal to the difference in the values of the integral J when taken along extremals of the set from L to 5 and from L to 6, and both of these are zero, since 5 and 6 lie on L. Then the portion of K which coincides with L may be replaced by the curve $\bar{K}_{56}$, which has only its end points on L, without altering the value of $J^*(K)$. It follows from the case above considered that $J^*(K) = J^*(K_1)$. From the preceding argument it is seen that, *at least for curves in the field which have a finite number of intersections or arcs in common with L, the integral J^* is independent of the path.*

Let $\bar{C}_{0'1}$ be any curve of class D' joining a point $0'$ of the curve L to the fixed point 1 and lying entirely in the field. Then, exactly as in § 6, the equation

$$J(\bar{C}_{0'1}) - J(C_{01}) = \int_{\bar{C}_{0'1}} E(x, y, z, \phi', \psi', \chi', x', y', z')\,dt$$

is derived by means of the invariant integral J^*. As has been seen before, this integral is always positive unless the comparison curve $\bar{C}_{0'1}$ coincides throughout with C_{01}, and the following theorem may be stated:

Let C_{01} be an arc of an extremal which joins a fixed curve L to a fixed point 1, *and which is cut transversally by the curve L at the point* 0. *Furthermore suppose that Legendre's condition for a minimum is satisfied along C_{01}. If then the determinant $\Delta(s, u, v)$, for the set of extremals to which L is transversal, vanishes only at the point* 0 *on C_{01}, this arc will minimize the integral J with respect to curves joining L with the point* 1.

THE SECOND DERIVATIVES OF THE EXTREMAL-INTEGRAL*

BY

ARNOLD DRESDEN

Introduction.

Suppose that an extremal has been found for the problem of minimizing the integral †

$$I = \int F(x, y, x', y')\,dt, \tag{1}$$

which passes through the points $A_1(a_1, b_1)$ and $A_2(a_2, b_2)$, along which $F_1 > 0$, and for which A_1 and A_2 are not conjugate points. Then it may be shown as in § 1 below that if $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are taken in a sufficiently small vicinity of A_1 and A_2 respectively, a unique extremal $\mathfrak{E}$ can be constructed passing through P_1 and P_2,‡ along which $F_1 > 0$ and for which P_1 and P_2 are not conjugate points. The integral (1) taken along P_1P_2 becomes a single-valued function of x_1, y_1, x_2 and y_2, uniquely defined for sufficiently small values of $|x_1 - a_1|$, $|y_1 - b_1|$, $|x_2 - a_2|$ and $|y_2 - b_2|$, which we denote by

$$\Im(x_1, y_1, x_2, y_2). \tag{2}$$

This function, commonly called the "*extremal-integral*," is identical with Hamilton's principal function. If the original extremal A_1A_2 actually furnishes a minimum for (1), then (2) must be a minimum in the ordinary sense for $x_1 = a_1$, $y_1 = b_1$, $x_2 = a_2$, $y_2 = b_2$. We are thus enabled to derive necessary conditions for a minimum of (1) by a discussion of (2) and its derivatives with respect to x_1, y_1, x_2 and y_2. §

The first derivatives of the function $\Im$ were given by HAMILTON in 1835. ‖

* Presented to the Society September 11, 1908. Received for publication June 15, 1908.

† The function F and the extremals are restricted by homogeneity and continuity conditions, for an explicit statement of which we refer the reader to BOLZA, *Vorlesungen über Variationsrechnung* (Leipzig, Teubner, 1908), pp. 193, 194. For terms and notations current in the Calculus of Variations and used here without explanation we refer to the same source.

‡ Compare BOLZA, loc. cit., § 37*a*.

§ This method of the Calculus of Variations, frequently called the "method of differentiation," was suggested by DIENGER in 1867. For further bibliographical reference see BOLZA, loc. cit., § 38.

‖ Philosophical Transactions of the Royal Society of London, 1835, part I, p. 99.

The object of the present paper is to obtain *explicit expressions for the second derivatives of the extremal-integral** (§ 1–§ 4), by means of which a simple determination of conjugate and focal points will be possible. In § 5 we treat by this method the problem of minimizing the integral (1) when *one end-point is movable along a fixed curve*, and in § 6 the same problem when *both end-points are movable*. Thus new proofs are given of the theorems first proved by BLISS.† In § 7 the same method is applied to the discussion of *conjugate points on discontinuous solutions*, previously investigated by CARATHEODORY‡ and BOLZA.§ The results arrived at are in appearance in *direct contradiction* with theirs. The discussion of this contradiction appears in § 8, where it is shown by means of a relation between the Weierstrass E-function and Caratheodory's invariant Ω_0, that the case in which the contradiction occurred cannot arise.

§ 1. *Construction of an extremal through two given points.*‖

We take Euler's differential equation, written by BLISS¶ in the following form:

$$(3)\qquad \frac{dx}{ds}=\cos\theta,\qquad \frac{dy}{ds}=\sin\theta,\qquad \frac{d\theta}{ds}=\frac{F_{x'y}-F_{xy'}}{F_1\left(\sqrt{x'^2+y'^2}\right)^3}.$$

Denoting by

$$(4)\qquad \begin{aligned} x&=\mathfrak{X}(s-s_i;\ x_i, y_i, \theta_i),\\ y&=\mathfrak{Y}(s-s_i;\ x_i, y_i, \theta_i),\\ \theta&=\Theta(s-s_i;\ x_i, y_i, \theta_i), \end{aligned}$$

that particular solution of (3) which satisfies the initial conditions

$$x=x_i,\ y=y_i,\ \theta=\theta_i$$

at $s=s_i$, we solve the system

$$(5)\qquad x_2=\mathfrak{X}(s_2-s_1;\ x_1, y_1, \theta_1),\qquad y_2=\mathfrak{Y}(s_2-s_1;\ x_1, y_1, \theta_1)$$

for s_2 and θ_1 as functions of x_1, y_1, x_2 and y_2, the value s_1 being chosen arbitrarily. This is always possible, according to the well-known theory of implicit

* In the Sächsische Berichte, 1883–1884, part II, p. 99, A. MAYER has used a similar method for the problem of variable end-points.

† See Transactions of the American Mathematical Society, vol. 3 (1902), p. 136, and Mathematische Annalen, vol. 58 (1903), p. 70.

‡ *Dissertation*, Göttingen, 1904, p. 31, and Mathematische Annalen, vol. 62 (1906), p. 449.

§ *Vorlesungen*, chap. VIII, and American Journal of Mathematics, vol. 30 (1908), p. 209.

‖ Compare BOLZA, *Vorlesungen*, § 37*a*). For notation, ibid., § 27*b*).

¶ Transactions of the American Mathematical Society, vol. 7 (1906), p. 188.

functions, if A_1 and A_2 are not conjugate.* When s_2 and θ_1 are so determined, *the functions*

$$
\begin{aligned}
x &= \mathfrak{X}(s - s_1;\ x_1, y_1, \theta_1) \equiv x(s),\\
y &= \mathfrak{Y}(s - s_1;\ x_1, y_1, \theta_1) \equiv y(s),\\
\theta &= \Theta(s - s_1;\ x_1, y_1, \theta_1)
\end{aligned} \tag{6}
$$

represent the required extremal $\mathfrak{E}$, and we have†

$$
\begin{aligned}
&\mathfrak{X}(0;\ x_1, y_1, \theta_1) = x_1, \quad \mathfrak{Y}(0;\ x_1, y_1, \theta_1) = y_1, \quad \Theta(0;\ x_1, y_1, \theta_1) = \theta_1,\\
&\mathfrak{X}(s_2 - s_1;\ x_1, y_1, \theta_1) = x_2, \quad \mathfrak{Y}(s_2 - s_1;\ x_1, y_1, \theta_1) = y_2, \quad \Theta(s_2 - s_1;\ x_1, y_1, \theta_1) = \theta_2.
\end{aligned} \tag{7}
$$

Further, we get from (3) and (7) the following useful identities:

$$
\begin{aligned}
\mathfrak{X}_s(0;\ x_1, y_1, \theta_1) &= \cos\theta_1, & \mathfrak{Y}_s(0;\ x_1, y_1, \theta_1) &= \sin\theta_1,\\
\mathfrak{X}_s(s_2 - s_1;\ x_1, y_1, \theta_1) &= \cos\theta_2, & \mathfrak{Y}_s(s_2 - s_1;\ x_1, y_1, \theta_1) &= \sin\theta_2,\\
\mathfrak{X}_{x_1}(0;\ x_1, y_1, \theta_1) &= 1, & \mathfrak{Y}_{x_1}(0;\ x_1, y_1, \theta_1) &= 0,\\
\mathfrak{X}_{y_1}(0;\ x_1, y_1, \theta_1) &= 0, & \mathfrak{Y}_{y_1}(0;\ x_1, y_1, \theta_1) &= 1,\\
\mathfrak{X}_{\theta_1}(0;\ x_1, y_1, \theta_1) &= 0, & \mathfrak{Y}_{\theta_1}(0;\ x_1, y_1, \theta_1) &= 0.
\end{aligned} \tag{8}
$$

It follows from (4) that *the extremal* $\mathfrak{E}$ *may also be represented in the form*

$$
\begin{aligned}
x &= \mathfrak{X}(s - s_2;\ x_2, y_2, \theta_2) \equiv x(s),\\
y &= \mathfrak{Y}(s - s_2;\ x_2, y_2, \theta_2) \equiv y(s),\\
\theta &= \Theta(s - s_2;\ x_2, y_2, \theta_2),
\end{aligned} \tag{9}
$$

from which we can derive formulas analogous to (7) and (8) and obtainable from them by interchange of the subscripts 1 and 2.‡ It is evident that the functions $x(s)$ and $y(s)$ as defined by (6) on the one hand, and by (9) on the other hand, are identical *in the variable* s for the range $s_1 \leqq s \leqq s_2$.

Whenever a distinction between these two forms of the extremal shall be necessary in the sequel, we shall use the following abbreviated notations:

$$
\begin{aligned}
&\mathfrak{X}(s - s_i;\ x_i, y_i, \theta_i) = \mathfrak{X}^i(s), && (i = 1, 2).\\
&\mathfrak{Y}(s - s_i;\ x_i, y_i, \theta_i) = \mathfrak{Y}^i(s),\\
&\cos\theta_i = p_i, \qquad \sin\theta_i = q_i.
\end{aligned} \tag{10}
$$

*Compare C. JORDAN, *Cours d'Analyse*, vol. 1, 2d ed., § 92. The theorem is applicable because equations (5) are satisfied by the tangential angle of A_1A_2 at A_1 and by the parameter value of A_2 on A_1A_2, if we substitute for x_1, y_1, and x_2, y_2 the coördinates of A_1 and A_2, and because furthermore the Jacobian does not vanish if A_1 and A_2 are not conjugate (see also § 2 and §4, and BOLZA, loc. cit., p. 234).

† The last of these equations is to be considered as defining θ_2.

‡ It is to be observed that s_2 and θ_2, used in (9), are not arbitrary, but are quantities defined by (5) and (7) respectively, whereas s_1 and θ_1 following from them are identical with the quantities defined by these same symbols in (5).

§ 2. *The extremal-integral.*

Along this extremal $\mathfrak{E}$ we compute now the integral (1), which furnishes us the function (2) in two forms:

$$\Im(x_1, y_1, x_2, y_2) = \int_{s_1}^{s_2} F[\mathfrak{x}^1(s), \mathfrak{y}^1(s), \mathfrak{x}^1_s(s), \mathfrak{y}^1_s(s)]\,ds$$

$$= \int_{s_1}^{s_2} F[\mathfrak{x}^2(s), \mathfrak{y}^2(s), \mathfrak{x}^2_s(s), \mathfrak{y}^2_s(s)]\,ds.$$

We find then the *first derivatives of the extremal-integral,*

$$(11)\qquad \begin{aligned} \frac{\partial \Im}{\partial x_1} &= -F_{x'}(x_1, y_1, p_1, q_1), & \frac{\partial \Im}{\partial x_2} &= F_{x'}(x_2, y_2, p_2, q_2), \\ \frac{\partial \Im}{\partial y_1} &= -F_{y'}(x_1, y_1, p_1, q_1), & \frac{\partial \Im}{\partial y_2} &= F_{y'}(x_2, y_2, p_2, q_2), \end{aligned}$$

which formulæ correspond to *Hamilton's first derivatives of the principal function.** For the determination of the second derivatives of (2) we have first to determine $\partial\theta_i/\partial z$, in which $i = 1, 2$, and z is any one of the 4 variables x_1, y_1, x_2, y_2. Differentiating (5) with respect to x_1, we obtain

$$0 = \mathfrak{x}^1_s(s_2)\frac{\partial s_2}{\partial x_1} + \mathfrak{x}^1_{x_1}(s_2) + \mathfrak{x}^1_{\theta_1}(s_2)\frac{\partial \theta_1}{\partial x_1},$$

$$0 = \mathfrak{y}^1_s(s_2)\frac{\partial s_2}{\partial x_1} + \mathfrak{y}^1_{x_1}(s_2) + \mathfrak{y}^1_{\theta_1}(s_2)\frac{\partial \theta_1}{\partial x_1}.$$

Similar equations are obtained by differentiating with respect to x_2, y_1 and y_2, and four more by interchange of subscripts and superscripts 1 and 2. All these equations are uniquely solvable† for $\partial\theta_i/\partial z$, so that *we obtain the following results:*

$$(12)\qquad \begin{aligned} \frac{\partial \theta_1}{\partial x_1} &= -\frac{\xi_1(s_2)}{u_1(s_2)}, & \frac{\partial \theta_2}{\partial x_1} &= -\frac{q_1}{u_2(s_1)}, \\ \frac{\partial \theta_1}{\partial y_1} &= -\frac{\eta_1(s_2)}{u_1(s_2)}, & \frac{\partial \theta_2}{\partial y_1} &= \frac{p_1}{u_2(s_1)}, \\ \frac{\partial \theta_1}{\partial x_2} &= -\frac{q_2}{u_1(s_2)}, & \frac{\partial \theta_2}{\partial x_2} &= -\frac{\xi_2(s_1)}{u_2(s_1)}, \\ \frac{\partial \theta_1}{\partial y_2} &= \frac{p_2}{u_1(s_2)}, & \frac{\partial \theta_2}{\partial y_2} &= -\frac{\eta_2(s_1)}{u_2(s_1)}, \end{aligned}$$

* HAMILTON, loc. cit.; compare also BOLZA, loc. cit., § 37*b*).

† Compare with the first footnote on page 469.

where

$$\begin{aligned}
\xi_i(s) &= \mathfrak{X}^i_s(s)\,\mathfrak{Y}^i_{x_i}(s) - \mathfrak{X}^i_{x_i}(s)\,\mathfrak{Y}^i_s(s),\\
\eta_i(s) &= \mathfrak{X}^i_s(s)\,\mathfrak{Y}^i_{y_i}(s) - \mathfrak{X}^i_{y_i}(s)\,\mathfrak{Y}^i_s(s), \qquad (i=1,2),\\
u_i(s) &= \mathfrak{X}^i_s(s)\,\mathfrak{Y}^i_{\theta_i}(s) - \mathfrak{X}^i_{\theta_i}(s)\,\mathfrak{Y}^i_s(s).
\end{aligned} \tag{13}$$

§ 3. *Particular solutions of Jacobi's differential equation.*

The functions ξ_i, η_i and u_i have the following properties:

1) They are particular solutions of Jacobi's differential equation for the extremal $\mathfrak{E}$,

$$F_2\omega - \frac{d}{ds}\left(F_1\frac{d\omega}{ds}\right) = 0,$$

the arguments of F_1 and F_2 being $x(s)$, $y(s)$, $x'(s)$, $y'(s)$. The proof of this statement can be given in precisely the same way as is usually followed for the proof of Jacobi's theorem: *

2) They satisfy the following conditions:

$$\begin{aligned}
&\xi_i(s_i) = -q_i, && \eta_i(s_i) = p_i, && u_i(s_i) = 0, \quad (i=1,2),\\
&\xi'_i(s_i) = -y''(s_i), && \eta'_i(s_i) = x''(s_i), && u'_i(s_i) = 1,
\end{aligned} \tag{14}$$

which follow from (13) by means of (3), (8) and interchange of subscripts 1 and 2.

For our further work *we introduce now also those particular solutions* $v_i(s)$ *of Jacobi's equation which satisfy the conditions*

$$v_i(s_i) = 1, \qquad v'_i(s_i) = 0. \tag{15}$$

It is clear that $u_1(s)$, $v_1(s)$ and $u_2(s)$, $v_2(s)$ are linearly independent solutions of that equation,† so that we can express $\xi_1(s)$, $\eta_1(s)$ and $\xi_2(s)$, $\eta_2(s)$ linearly in terms of $u_1(s)$, $v_1(s)$ and $u_2(s)$, $v_2(s)$ respectively.‡ We find, using (14) and (15), that

$$\xi_i(s) = -y''(s_i)u_i(s) - q_i v_i(s), \qquad \eta_i(s) = x''(s_i)u_i(s) + p_i v_i(s). \tag{16}$$

Using (16), we can now transform (12) and we obtain *the following formulæ which express the partial derivatives of the tangential angles of the extremal* $\mathfrak{E}$ *at* P_1 *and* P_2 *in terms of two sets of two linearly independent integrals of Jacobi's differential equation for that extremal:*

* Compare BOLZA, loc. cit., § 12*b*.

† Compare C. JORDAN, *Cours d'Analyse*, 2d ed., vol. 3, § 122.

‡ Ibid., § 119.

$$
(17)\qquad
\begin{aligned}
&\frac{\partial\theta_1}{\partial x_1} = y''(s_1) + \frac{q_1 v_1(s_2)}{u_1(s_2)}, && \frac{\partial\theta_2}{\partial x_1} = -\frac{q_1}{u_2(s_1)},\\
&\frac{\partial\theta_1}{\partial y_1} = -x''(s_1) - \frac{p_1 v_1(s_2)}{u_1(s_2)}, && \frac{\partial\theta_2}{\partial y_1} = \frac{p_1}{u_2(s_1)},\\
&\frac{\partial\theta_1}{\partial x_2} = -\frac{q_2}{u_1(s_2)}, && \frac{\partial\theta_2}{\partial x_2} = y''(s_2) + \frac{q_2 v_2(s_1)}{u_2(s_1)},\\
&\frac{\partial\theta_1}{\partial y_2} = \frac{p_2}{u_1(s_2)}, && \frac{\partial\theta_2}{\partial y_2} = -x''(s_2) - \frac{p_2 v_2(s_1)}{u_2(s_1)}.
\end{aligned}
$$

§ 4. *The second derivatives of the extremal-integral.*

We can now at once determine the second derivatives of (2) with respect to any two of the variables x_1, y_1, x_2, y_2. Differentiating (11) and remembering that

$$\frac{\partial p_i}{\partial z} = -q_i\frac{\partial\theta_i}{\partial z}, \qquad \frac{\partial q_i}{\partial z} = p_i\frac{\partial\theta_i}{\partial z},$$

we obtain the following

THEOREM. *The second derivatives of the extremal integral are given by the table*

(18)

	$\frac{\partial\mathfrak{J}}{\partial x_1}$	$\frac{\partial\mathfrak{J}}{\partial y_1}$	$\frac{\partial\mathfrak{J}}{\partial x_2}$	$\frac{\partial\mathfrak{J}}{\partial y_2}$
$\frac{\partial}{\partial x_1}$	$-L(s_1) + F_1(s)\frac{q_1^2 v_1(s_2)}{u_1(s_2)}$	$-M(s_1) - F_1(s_1)\frac{p_1 q_1 v_1(s_2)}{u_1(s_2)}$	$-F_1(s_1)\frac{q_1 q_2}{u_1(s_2)}$	$F_1(s_1)\frac{q_1 p_2}{u_1(s_2)}$
$\frac{\partial}{\partial y_1}$	$-M(s_1) - F_1(s_1)\frac{p_1 q_1 v_1(s_2)}{u_1(s_2)}$	$-N(s_1) + F_1(s_1)\frac{p_1^2 v_1(s_2)}{u_1(s_2)}$	$F_1(s_1)\frac{p_1 q_2}{u_1(s_2)}$	$-F_1(s_1)\frac{p_1 p_2}{u_1(s_2)}$
$\frac{\partial}{\partial x_2}$	$F_1(s_2)\frac{q_1 q_2}{u_2(s_1)}$	$-F_1(s_2)\frac{p_1 q_2}{u_2(s_1)}$	$L(s_2) - F_1(s_2)\frac{q_2^2 v_2(s_1)}{u_2(s_1)}$	$M(s_2) + F_1(s_2)\frac{p_2 q_2 v_2(s_1)}{u_2(s_1)}$
$\frac{\partial}{\partial y_2}$	$-F_1(s_2)\frac{q_1 p_2}{u_2(s_1)}$	$F_1(s_2)\frac{p_1 p_2}{u_2(s_1)}$	$M(s_2) + F_1(s_2)\frac{p_2 q_2 v_2(s_1)}{u_2(s_1)}$	$N(s_2) - F_1(s_2)\frac{p_2^2 v_2(s_1)}{u_2(s_1)}$

in which the functions F_1, L, M, N have the same meaning as in the Weierstrass theory, and in which the functions $u_1(s)$, $v_1(s)$, $u_2(s)$ and $v_2(s)$ are defined by (14) *and* (15), *and p_1, q_1, p_2, q_2 by* (10).

In order to show that these formulæ are independent of the order in which the two differentiations are performed, it is sufficient to prove

$$
(19)\qquad \frac{F_1(s_1)}{u_1(s_2)} + \frac{F_1(s_2)}{u_2(s_1)} = 0.
$$

We know that $u_2(s_2) = 0$, and since P_2 and P_1 are not conjugate, that $u_1(s_2) \neq 0$. Accordingly $u_1(s)$ and $u_2(s)$ are linearly independent solutions of Jacobi's equation, which we now write in the form:

$$\omega'' - \frac{F_1'}{F_1}\omega' + \frac{F_2}{F_1}\omega = 0.$$

Hence, by Abel's theorem,*

$$u_1(s)u_2'(s) - u_2(s)u_1'(s) = \frac{C}{F_1(s)}.$$

By applying (14) for $i = 1$ and $i = 2$, we obtain

$$-u_2(s_1) = \frac{C}{F_1(s_1)}, \qquad u_1(s_2) = \frac{C}{F_1(s_2)},$$

from which (19) follows immediately.

We conclude this paragraph by *establishing a relation between the functions* $u_i(s)$, $v_i(s)$ *and the Weierstrassian function* $\Theta(s, s_i)$ which is in current use in the literature. If $\vartheta_1(s)$ and $\vartheta_2(s)$ are any two linearly independent solutions of Jacobi's equation, we have†

$$\Theta(s, s_i) = \vartheta_1(s)\vartheta_2(s_i) - \vartheta_1(s_i)\vartheta_2(s).$$

Hence

$$\Theta(s_i, s_i) = 0,$$

$$\frac{\partial}{\partial s}\Theta(s, s_i)\Big|_{s=s_i} = \vartheta_1'(s_i)\vartheta_2(s_i) - \vartheta_1(s_i)\vartheta_2'(s_i),$$

$$\frac{\partial}{\partial s_i}\Theta(s, s_i)\Big|_{s=s_i} = \vartheta_1(s_i)\vartheta_2'(s_i) - \vartheta_1'(s_i)\vartheta_2(s_i),$$

$$\frac{\partial^2}{\partial s\,\partial s_i}\Theta(s, s_i)\Big|_{s=s_i} = \vartheta_1'(s_i)\vartheta_2'(s_i) - \vartheta_1'(s_i)\vartheta_2'(s_i).$$

Writing now

$$D(s) = \vartheta_1(s)\vartheta_2'(s) - \vartheta_1'(s)\vartheta_2(s),‡$$

$$u_i(s) = -\frac{\Theta(s, s_i)}{D(s_i)}, \qquad v_i(s) = \frac{\dfrac{\partial}{\partial s_i}\Theta(s, s_i)}{D(s_i)}, \tag{20}$$

we have

$$u_i(s_i) = 0, \qquad v_i(s_i) = 1,$$

$$u_i'(s_i) = 1, \qquad v_i'(s_i) = 0.$$

The functions $u_i(s)$ and $v_i(s)$ being uniquely determined by (14) and (15), it follows that those defined in (20) are identical with them.

* Compare STURM, *Cours d' Analyse*, 12th edition, vol. 2, § 609.

† Compare BOLZA, loc. cit., p. 233.

‡ Ibid.

§ 5. *The case of one variable end-point.*

We suppose that we have found an extremal $\mathfrak{E}$ which actually furnishes a minimum for (1) when the first end-point P_1 is movable along the fixed curve $\mathfrak{C}$ defined by the equations

$$\mathfrak{C}: \qquad x = \tilde{x}(a), \qquad y = \tilde{y}(a) \qquad (a_0 \leqq a \leqq a_2),$$

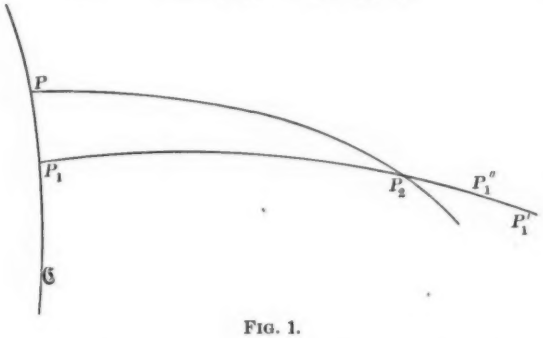

FIG. 1.

the second end-point P_2 being fixed. If a point P is taken on $\mathfrak{C}$ sufficiently near P_1, the construction of the unique extremal PP_2 can be carried out as described in § 1 and the extremal-integral can be computed along PP_2. *This extremal-integral becomes now a function of the parameter a of the point P,*

$$\mathfrak{J}[\tilde{x}(a), \tilde{y}(a), x_2, y_2] = \mathfrak{J}[a],$$

and must be a minimum for $a = a_1$, *if* P_1P_2 *actually minimizes* (1). Hence the necessary conditions for a minimum of (1) are in this case

$$\mathfrak{J}'[a_1] = 0 \tag{21}$$

and

$$\mathfrak{J}''[a_1] \geqq 0. \tag{22}$$

Making use of (11) and writing

$$\tilde{p}_1 = \frac{\tilde{x}'(a_1)}{\sqrt{\tilde{x}'^2(a_1) + \tilde{y}'^2(a_1)}}, \qquad \tilde{q}_1 = \frac{\tilde{y}'(a_1)}{\sqrt{\tilde{x}'^2(a_1) + \tilde{y}'^2(a_1)}}, \tag{23}$$

we find *

$$\mathfrak{J}'[a_1] = -F_{x'}(x_1, y_1, p_1, q_1)\tilde{p}_1 - F_{y'}(x_1, y_1, p_1, q_1)\tilde{q}_1;$$

this shows that (21) is nothing but the well-known *transversality-condition.* †

* BOLZA, loc. cit., § 38.

† Ibid., § 36.

Further, with the help of (18),

$$\mathfrak{J}''[a_1] = \tilde{p}_1^2[-L(s_1)+q_1^2F_1(s_1)Z_1(s_2)]+2\tilde{p}_1\tilde{q}_1[-M(s_1)-p_1q_1F_1(s_1)Z_1(s_2)]$$
$$+\tilde{q}_1^2[-N(s_1)+p_1^2F_1(s_1)Z_1(s_2)]-\tilde{x}_1''F_{x'}(s_1)-\tilde{y}_1''F_{y'}(s_1),$$

where

$$\tilde{x}_1'' = \tilde{x}''(s_1), \qquad \tilde{y}_1'' = \tilde{y}''(s_1)$$

and

$$Z_i(s) = \frac{v_i(s)}{u_i(s)}, \qquad (i=1,2). \tag{24}$$

Introducing further the abbreviations

$$\begin{aligned} A_1 &= \tilde{p}_1^2L(s_1)+2\tilde{p}_1\tilde{q}_1M(s_1)+\tilde{q}_1^2N(s_1)+\tilde{x}_1''F_{x'}(s_1)+\tilde{y}_1''F_{y'}(s_1)\\ B_1 &= F_1(s_1)(\tilde{p}_1q_1-p_1\tilde{q}_1)^2, \end{aligned} \tag{25}$$

both A_1 and B_1 being constants depending upon the curve $\mathfrak{C}$ and the point P_1, we obtain the following formula:*

$$\mathfrak{J}''[a_1] = -A_1+B_1Z_1(s_2).$$

A further necessary condition for a minimum is therefore

$$-A_1+B_1Z_1(s_2)\geqq 0,$$

or, since $B_1>0$,†

$$Z_1(s_2)\geqq\frac{A_1}{B_1}.$$

We have defined by equation (24),

$$Z_1(s) = \frac{v_1(s)}{u_1(s)}.$$

Hence

$$Z_1'(s) = \frac{u_1(s)v_1'(s)-v_1(s)u_1'(s)}{u_1^2(s)}.$$

But $u_1(s)$ and $v_1(s)$ being linearly independent solutions of Jacobi's equation,‡ we have by Abel's theorem,§

$$u_1(s)v_1'(s)-v_1(s)u_1'(s) = \frac{k}{F_1(s)}.$$

By means of (14) and (15) we find

$$u_1(s_1)v_1'(s_1)-v_1(s_1)u_1'(s_1) = -1.$$

* Compare BOLZA, loc. cit., § 39; BLISS, Transactions of the American Mathematical Society, vol. 3 (1902), p. 136.

† Leaving aside the case that $\mathfrak{E}$ and $\mathfrak{C}$ are tangent at P_1.

‡ See § 3.

§ See the first footnote on p. 473.

Therefore

$$k = -F_1(s_1),$$

and

$$u_1(s)v_1'(s) - v_1(s)u_1'(s) = -\frac{F_1(s_1)}{F_1(s)}. \tag{26}$$

Hence

$$Z_1'(s) = -\frac{F_1(s_1)}{F_1(s)u_1^2(s)} < 0. \tag{27}$$

Further, it is evident that

$$\lim_{s \doteq s_1 + 0} Z_1(s) = +\infty,$$

and hence by Sturm's theorem,*

$$\lim_{s \doteq s_1' - 0} Z_1(s) = -\infty,$$

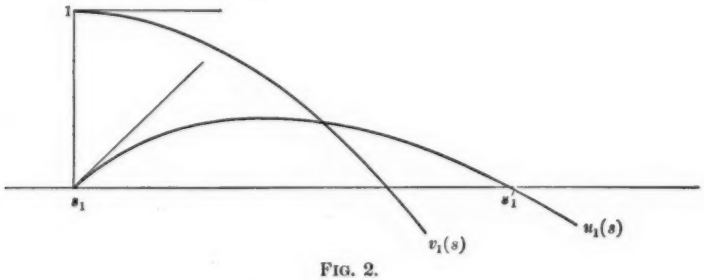

FIG. 2.

s_1' being the parameter value of the conjugate point of P_1 on $\mathfrak{E}$. We conclude that $Z_1(s)$ *is a monotonic decreasing function, taking every real value once and but once, as* s *increases from* s_1 *to* s_1'. Consequently there must be one real value of s between s_1 and s_1' for which

$$Z_1(s) = \frac{A_1}{B_1}.$$

Denoting this value by s_1'', it follows from (27) that *we must have*

$$s_2 \leqq s_1'', \tag{28}$$

in order that condition (22) *may be fulfilled.*

Thus we have given a new proof for Bliss's condition. BLISS† has investigated the geometrical meaning of the point determined on $\mathfrak{E}$ by s_1''. The properties of this so-called focal point have also been discussed by BOLZA.‡

* See STURM, *Cours d'Analyse*, 12th ed., vol. 2, no. 609.

† Transactions of the American Mathematical Society, vol. 3 (1902), p. 136.

‡ Loc. cit., § 39c.

§ 6. *The case of two variable end-points.*

In the same manner we treat now the case in which both end-points P_1 and P_2 are movable along two fixed curves, $\bar{\mathfrak{C}}$ and $\bar{\bar{\mathfrak{C}}}$ respectively. Let these curves be represented by

$$\bar{\mathfrak{C}}: \quad x = \tilde{x}(a), \quad y = \tilde{y}(a) \qquad (a_0 \leqq a \leqq a_1),$$

and

$$\bar{\bar{\mathfrak{C}}}: \quad x = \breve{x}(b), \quad y = \breve{y}(b) \qquad (b_0 \leqq b \leqq b_1),$$

and let us suppose that we had found an extremal $\mathfrak{E}$ actually furnishing a minimum for (1). Then if $P(a)$ and $Q(b)$ are taken sufficiently near P_1 and P_2

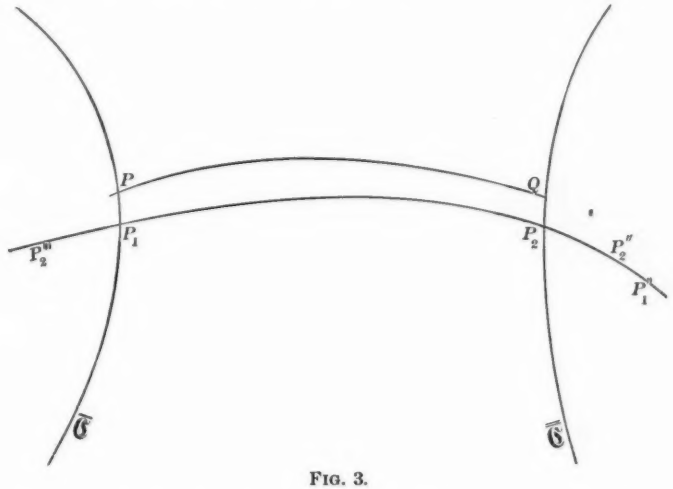

FIG. 3.

on $\bar{\mathfrak{C}}$ and $\bar{\bar{\mathfrak{C}}}$ respectively, the unique extremal PQ can be constructed and *the extremal-integral obtained now is a function of a and b*,

$$\mathfrak{J}[\tilde{x}(a), \tilde{y}(a), \breve{x}(b), \breve{y}(b)] = \mathfrak{J}[a, b].$$

This function must be a minimum for $a = a_1$, $b = b_2$, if P_1P_2 actually minimizes (1).

The necessary conditions for $\mathfrak{J}(a_2, b_2)$ *to be a minimum are*

$$\frac{\partial \mathfrak{J}[a, b]}{\partial a} = 0, \qquad \frac{\partial \mathfrak{J}[a, b]}{\partial b} = 0, \tag{29}$$

$$\frac{\partial^2 \mathfrak{J}[a, b]}{\partial a^2}\xi^2 + 2\frac{\partial^2 \mathfrak{J}[a, b]}{\partial a \partial b}\xi\eta + \frac{\partial^2 \mathfrak{J}[a, b]}{\partial b^2}\eta^2 \geqq 0, \tag{30}$$

for all real values of ξ *and* η.*

*See C. JORDAN, *Cours d'Analyse*, 2d ed., vol. 1, §§ 395–401.

As in § 5, equations (29) lead to the *transversality-conditions*

$$F_{x'}(x_1, y_1, p_1, q_1)\tilde{p}_1 + F_{y'}(x_1, y_1, p_1, q_1)\tilde{q}_1 = 0,$$
$$F_{x'}(x_2, y_2, p_2, q_2)\breve{p}_2 + F_{y'}(x_2, y_2, p_2, q_2)\breve{q}_2 = 0,$$

$\breve{p}_1$ and $\breve{q}_2$ being defined by formulæ analogous to (23).

Further, proceeding as in § 5, we find

$$\frac{\partial^2 \mathfrak{J}}{\partial a^2} = -A_1 + B_1 Z_1(s_2),$$

$$\frac{\partial^2 \mathfrak{J}}{\partial b^2} = A_2 - B_2 Z_2(s_1),$$

(31)

$$\frac{\partial^2 \mathfrak{J}}{\partial a\,\partial b} = F_1(s_2)\frac{(p_1\tilde{q}_1 - \tilde{p}_1 q_1)(p_2\breve{q}_2 - \breve{p}_2 q_2)}{u_2(s_1)}$$
$$= -F_1(s_1)\frac{(p_1\tilde{q}_1 - \tilde{p}_1 q_1)(p_2\breve{q}_2 - \breve{p}_2 q_2)}{u_1(s_2)},$$

where the last two expressions are equivalent on account of (19); where also A_1, B_1, $Z_1(s)$ are defined by (25) and (24), and A_2, B_2, $Z_2(s)$ by analogous formulæ, obtained from them by a change of index.

Condition (30) will be fulfilled if*

(32) $$\frac{\partial^2 \mathfrak{J}}{\partial a^2} \geqq 0$$

and

(33) $$\frac{\partial^2 \mathfrak{J}}{\partial a^2}\frac{\partial^2 \mathfrak{J}}{\partial b^2} - \left(\frac{\partial^2 \mathfrak{J}}{\partial a\,\partial b}\right)^2 \geqq 0,$$

from which follows

(32a) $$\frac{\partial^2 \mathfrak{J}}{\partial b^2} \geqq 0.$$

As in § 5, the relation (32) leads at once to the inequality

(28) $$s_2 \leqq s_1''.$$

In the same manner we could show that (32a) leads to the relation

$$s_2''' \leqq s_1,$$

s_2''' being defined by the equation

(34) $$Z_2(s) = \frac{A_2}{B_2} \qquad (s < s_1).$$

It can furthermore easily be shown that there is also a value of s beyond s_2, for which (34) is fulfilled. This value we denote by s_2''.

For the further discussion of (33) we introduce now s_1'' and s_2'' by means of the relations

$$A_1 = B_1 Z_1(s_1''), \qquad A_2 = B_2 Z_2(s_2'').$$

* See C. JORDAN, ibid.

Then (31) becomes

$$\frac{\partial^2 \mathfrak{J}}{\partial a^2} = B_1 \{ Z_1(s_2) - Z_1(s_1'') \},$$

$$\frac{\partial^2 \mathfrak{J}}{\partial b^2} = B_2 \{ Z_2(s_2'') - Z_2(s_1) \}, \tag{35}$$

$$\left(\frac{\partial^2 \mathfrak{J}}{\partial a \partial b} \right)^2 = - \frac{B_1 B_2}{u_1(s_2) u_2(s_1)}.$$

We desire to express the functions of s with subscript 2 *in terms of those with subscript* 1. For this purpose we write

$$u_2(s) = \alpha_{11} u_1(s) + \alpha_{12} v_1(s), \qquad v_2(s) = \alpha_{21} u_1(s) + \alpha_{22} v_1(s).$$

Then using (14) and (15) we can determine α_{ij}, and we find by making use of (26),

$$u_2(s) = \frac{F_1(s_2)}{F_1(s_1)} \{ u_1(s) v_1(s_2) - u_1(s_2) v_1(s) \},$$

$$v_2(s) = - \frac{F_1(s_2)}{F_1(s_1)} \{ u_1(s) v_1'(s_2) - u_1'(s_2) v_1(s) \}. \tag{36}$$

Consequently

$$Z_2(s) = - \frac{u_1(s) v_1'(s_2) - u_1'(s_2) v_1(s)}{u_1(s) v_1(s_2) - u_1(s_2) v_1(s)},$$

and in particular

$$Z_2(s_1) = - \frac{u_1'(s_2)}{u_1(s_2)}, \qquad Z_2(s_2'') = - \frac{v_1'(s_2) - u_1'(s_2) Z_1(s_2'')}{v_1(s_2) - u_1(s_2) Z_1(s_2'')}.$$

From (36) or from (19) it follows that

$$u_2(s_1) = - \frac{F_1(s_2)}{F_1(s_1)} u_1(s_2).$$

We can also write

$$\frac{\partial^2 \mathfrak{J}}{\partial b^2} = B_2 \frac{F_1(s_1)}{F_1(s_2) u_1(s_2) \{ v_1(s_2) - u_1(s_2) Z_1(s_2'') \}}.$$

Hence, since B_1, B_2, $F_1(s_1)$, $F_1(s_2)$ are all > 0, the relation (33) becomes

$$\frac{Z_1(s_2) - Z_1(s_1'')}{u_1(s_2) \{ v_1(s_2) - u_1(s_2) Z_1(s_2'') \}} - \frac{1}{u_1^2(s_2)} \geqq 0$$

or

$$\frac{Z_1(s_2'') - Z_1(s_1'')}{Z_1(s_2) - Z_1(s_2'')} \geqq 0.$$

Since s_2'' was by definition beyond s_2, *this leads, in view of the relation* (27), *to the condition*

$$s_2'' < s_1'',$$

a result which has previously been obtained by BLISS,* who also pointed out its geometrical interpretation.

§ 7. *Discontinuous solutions.*

We propose next to investigate under what conditions a curve which has a finite discontinuity in its slope, a so-called discontinuous solution, may minimize the integral (1). We suppose that we have a broken curve $P_1P_0P_2$ actually minimizing (1). We know then from the current theory that each one of its branches must be an extremal along which Legendre's and Jacobi's conditions for ordinary extremals must be satisfied.† Taking now a point P sufficiently near P_0, we can construct uniquely the extremals P_1P and PP_2, the first one by identifying P with the point P_2 of § 1, the second one by identifying

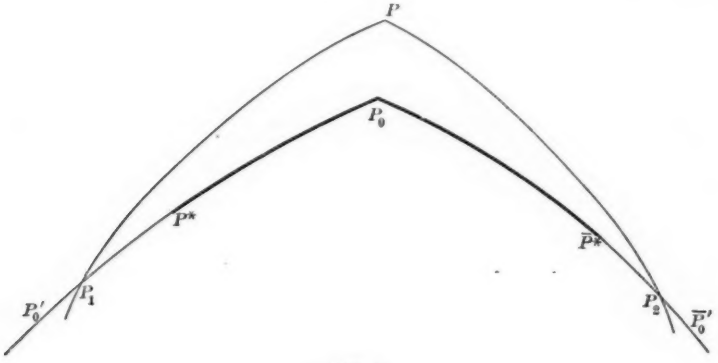

FIG. 4.

P with the point P_1 of § 1. Computing (1) along each of these extremals, we obtain the extremal-integral along the broken curve P_1PP_2 as a function of the coördinates x, y of P:

$$\mathfrak{J}(x_1, y_1, x, y) + \mathfrak{J}(x, y, x_2, y_2) = \mathfrak{J}\{x, y\}.$$

This function is to be a minimum for $x = x_0$, $y = y_0$, if $P_1P_0P_2$ is actually to furnish a minimum for (1).

We suppose that $P_1P_0P_2$ is represented in the form

$$x = x(s), \qquad y = y(s), \qquad s_1 \leqq s \leqq s_0,$$

$$x = \bar{x}(s), \qquad y = \bar{y}(s), \qquad s_0 \leq s \leq s_2,$$

* Mathematische Annalen, vol. 58 (1904), p. 74.

† Compare BOLZA, loc. cit., § 48*a*.

and introduce the notation

$$\phi[x(s), y(s), x'(s), y'(s)] \equiv \phi(s), \quad \phi[\bar{x}(s), \bar{y}(s), \bar{x}'(s), \bar{y}'(s)] \equiv \bar{\phi}(s),$$

ϕ being any function of x, y, x', y', subject to the ordinary continuity-restrictions.

In order to be able to apply the results of §§ 1–4, we must identify, throughout the discussion, P_0 with P_2 of § 1 when we consider P_0 as a point of P_1P_0, and P_0 with P_1 of § 1 when we consider P_0 as a point of P_0P_2. So, for instance, $u_2(s)$ goes over into $u_0(s)$, whereas $u_1(s)$ becomes $\bar{u}_0(s)$, etc.

With this agreement, we proceed to establish necessary conditions for a minimum of $\mathfrak{J}\{x, y\}$. *These necessary conditions are derived from the well-known relations*

$$\frac{\partial \mathfrak{J}\{x_0, y_0\}}{\partial x} = 0, \qquad \frac{\partial \mathfrak{J}\{x_0, y_0\}}{\partial y} = 0, \tag{37}$$

and

$$\frac{\partial^2 \mathfrak{J}\{x_0, y_0\}}{\partial x^2}\xi^2 + 2\frac{\partial^2 \mathfrak{J}\{x_0, y_0\}}{\partial x \partial y}\xi\eta + \frac{\partial^2 \mathfrak{J}\{x_0, y_0\}}{\partial y^2}\eta^2 \geqq 0. \tag{38}$$

which must hold for all real values of ξ *and* η.

We obtain, by using (11),

$$\frac{\partial \mathfrak{J}\{x_0, y_0\}}{\partial y} = F_{x'}(s_0) - \bar{F}_{x'}(s_0), \qquad \frac{\partial \mathfrak{J}\{x_0, y_0\}}{\partial y} = F_{y'}(s_0) - \bar{F}_{y'}(s_0),$$

whence we conclude that (37) *is identical with the Erdmann-Weierstrass corner condition.* *

Further, by using (18), the condition (38) becomes

$$-(A_0\xi^2 + 2B_0\xi\eta + C_0\eta^2) + \bar{F}_1(s_0)\bar{Z}_0(s_2)(\xi\bar{q}_0 - \eta\bar{p}_0)^2 - F_1(s_0)Z_0(s_1)(\xi q_0 - \eta p_0)^2 \geqq 0,$$

where

$$A_0 = \bar{L}(s_0) - L(s_0), \qquad B_0 = \bar{M}(s_0) - M(s_0), \qquad C_0 = \bar{N}(s_0) - N(s_0).$$

By means of the transformation

$$\xi = p_0\xi' - \bar{p}_0\eta', \qquad \eta = q_0\xi' - \bar{q}_0\eta', \tag{39}$$

the above homogeneous quadratic form goes over into

$$P_0(s_2)\xi'^2 + 2\Omega_0\xi'\eta' + R_0(s_1)\eta'^2,$$

where

$$P_0(s_2) = -T_0 + \bar{F}_1(s_0)\bar{Z}_0(s_2)(p_0\bar{q}_0 - \bar{p}_0q_0)^2,$$

$$R_0(s_1) = -\bar{T}_0 - F_1(s_0)Z_0(s_1)(p_0\bar{q}_0 - \bar{p}_0q_0)^2.$$

* Compare BOLZA, loc. cit., § 48*b*.

$$\Omega_0 = A_0 p_0 \bar{p}_0 + 2B_0(p_0\bar{q}_0 + \bar{p}_0 q_0) + C_0 q_0 \bar{q}_0, \tag{40}$$

$$T_0 = A_0 p_0^2 + 2B_0 p_0 q_0 + C_0 q_0^2,$$

$$\overline{T}_0 = A_0 \bar{p}_0^2 + 2B_0 \bar{p}_0 \bar{q}_0 + C_0 \bar{q}_0^2.$$

In passing, we notice the relation

$$\Omega_0^2 = T_0 \overline{T}_0 - (A_0 C_0 - B_0^2)(p_0\bar{q}_0 - \bar{p}_0 q_0)^2. \tag{41}$$

Moreover the determinant $p_0\bar{q}_0 - \bar{p}_0 q_0$ of the system (39) is not zero if

$$\bar{\theta}_0 - \theta_0 \not\equiv 0 \pmod{2\pi},$$

i. e., if there is a corner at P_0. *The conditions for a minimum may therefore be stated as follows:* †

I. *If* $\Omega_0 = 0$, *necessary conditions are*

$$P_0(s_2) \geqq 0, \tag{42}$$

$$R_0(s_1) \geqq 0. \tag{43}$$

II. *If* $\Omega_0 \neq 0$, *we have as necessary conditions*

$$P_0(s_2) > 0, \tag{42a}$$

$$P_0(s_2) R_0(s_1) - \Omega_0^2 \geqq 0. \tag{44}$$

Conditions (42a) and (44) have as a consequence

$$R_0(s_1) > 0. \tag{43a}$$

I. Since $\overline{Z}_0(s)$ is identical with $Z_1(s)$ of § 5, we conclude immediately that *there exists between* s_0 *and* $\bar{s}_0'$ *one and only one value* $\bar{s}^*$ *for which* $P_0(s) = 0$, ‡ *and that a necessary condition for a minimum is*

$$s_2 \leqq \bar{s}^*. \tag{45}$$

The equation $P_0(s) = 0$ *satisfied by* $\bar{s}^*$ *can be written in the form*

$$\bar{Z}_0(\bar{s}^*) = \frac{T_0}{\overline{F}_1(\bar{s}_0)(p_0\bar{q}_0 - \bar{p}_0 q_0)^2}. \tag{46}$$

In the same manner, *the relation* (43) *leads to the necessary condition*

$$s_1 \geqq s^*, \tag{47}$$

* This function Ω_0 is the invariant, introduced by CARATHEODORY in a different form (see *Dissertation*, p. 31 and Mathematische Annalen, vol. 62 (1906), p. 473), to which the present one can be easily reduced by means of the relations between L, M, N and the derivatives of F, as well as the homogeneity properties of F. Compare BOLZA, loc. cit., § 49, and American Journal of Mathematics, vol. 30 (1908), pp. 212 and 214.

† Compare footnote on page 477.

‡ The symbol $\bar{s}_0'$ denotes the parameter value conjugate to s_0 on P_0P_2, i. e., the root of $\bar{u}_0(s) = 0$ which follows next after s_0.

s^* being defined by the equation

$$Z_0(s^*) = \frac{-\overline{T}_0}{F_1(s_0)(p_0\bar{q}_0 - \bar{p}_0 q_0)^2}. \tag{48}$$

The parameter values s^* and $\bar{s}^*$ define points P^* and $\bar{P}^*$ on P_1P_0 and P_0P_2 respectively (see Fig. 4). Our result is, then, *that the end-points of a minimizing "broken extremal" must lie on the arc bounded by these two critical points.**

II. In the first place, *conditions* (45) *and* (47) *must be fulfilled in the stronger forms*

$$s_2 < \bar{s}^*, \tag{45a}$$

and

$$s_1 > s^*. \tag{47a}$$

Secondly, from the properties of $Z_0(s)$ and $\overline{Z}_0(s)$ it follows that if we consider P_1 as fixed, a point $\overline{P}_1$ is uniquely determined on P_0P_2; or if we consider P_2 as fixed, a point $\overline{P}_2$ is uniquely determined on P_1P_0, by the relation

$$P_0(s_2)R_0(s_1) - \Omega_0^2 = 0. \tag{49}$$

Furthermore, in order to have (44) *fulfilled, we must have*

$$s_2 \leqq \bar{s}_1 \,(\text{or } s_1 \geqq \bar{s}_2), \tag{50}$$

where $\bar{s}_1$, and $\bar{s}_2$ are the parameter values of $\overline{P}_1$ and $\overline{P}_2$ respectively.†

Summarizing, we obtain for this case the following

THEOREM. *If the inequality*

$$\Omega_0 = A_0p_0\bar{p}_0 + B_0(p_0\bar{q}_0 + \bar{p}_0q_0) + C_0q_0\bar{q}_0 \neq 0$$

holds, then necessary conditions for a minimum of the integral (1) *are*

$$s_1 > s^* \qquad \text{and} \qquad s_0 \leqq s_2 < \bar{s}_1;$$

if $\Omega_0 = 0$, it is necessary that

$$s_1 \geqq s^* \qquad \text{and} \qquad s_2 \leqq \bar{s}^*.$$

The relation (49) connecting s_1 with $\bar{s}_1$ (and s_2 with $\bar{s}_2$) may be written in explicit form by means of (40) and (41). We find that

$$\begin{aligned} &A_0C_0 - B_0^2 + F_1(s_0)(A_0p_0^2 + 2B_0p_0q_0 + C_0q_0^2)Z_0(s_1) \\ &\quad - \overline{F}_1(s_0)(A_0\bar{p}_0^2 + 2B_0\bar{p}_0\bar{q}_0 + C_0\bar{q}_0^2)\overline{Z}_0(\bar{s}_1) \\ &\quad - F_1(s_0)\overline{F}_1(s_0)(p_0\bar{q}_0 - \bar{p}_0q_0)^2Z_0(s_1)\overline{Z}_0(\bar{s}_1) = 0. \end{aligned} \tag{49a}$$

The relation occurs in this form in Bolza's work.‡

* For the geometrical interpretation of these points, see BOLZA, loc. cit., § 49*a*, and American Journal of Mathematics, vol. 30 (1908), p. 217; also CARATHEODORY, *Dissertation*, p. 31.

† For the geometrical interpretation of $\overline{P}_1$ and $\overline{P}_2$, see ibid.

‡ See the first footnote on p. 482.

We see that the point $\overline{P}_1$ plays the rôle in the theory of discontinuous extremals which the conjugate point plays in the theory of continuous extremals. For this reason $\overline{P}_1$ is called the *conjugate point* of P_1 on P_0P_2.

In order to show that conditions (47a) and (50) have (45a) as their consequence, we must prove that

$$s_0 < s_1 < s^*, \quad \text{if} \quad s_0 > s_1 > s^*. \tag{51}$$

Indicating the functional dependence of s_1 on s_1 by means of the equation

$$s_1 = s_2(s_1),$$

which is implicitly contained in (49a), we can easily show by means of (46) and (48) that

$$s_2(s^*) = s_0, \qquad s_2(s_0) = s^*,$$

and furthermore, by using (41) and (27), that

$$s_2'(s_1) = \frac{F_1(s_0)Z_0'(s_1)\Omega_0^2}{\overline{F}(s_0)\overline{Z}_0'[s_2(s_1)]R_0^2(s_1)} > 0,$$

from which the inequalities (51) follow immediately.

§ 8. *Contradiction with previous results.*

The theorem stated in the preceding paragraph is in direct contradiction with results previously obtained by CARATHEODORY * and BOLZA,† who give *sufficient conditions for a minimum, less restricting than the necessary conditions arrived at here.* We shall briefly state the contradiction.

According to BOLZA any one of the combinations (AIa, B), (AIb, B), (AII, B) from the following set are sufficient conditions for a minimum:

$$A.\ \text{I}a\colon\ \Omega_0 > 0, \quad P_0' < P_1 < P^*, \quad P_0 < P_2 < \overline{P}_1;$$

$$\text{I}b\colon\ \Omega_0 > 0, \quad P^* < P_1 < P_0, \quad P_0 < P_2 < \overline{P}_0';$$

$$\text{II}\colon\ \Omega_0 < 0, \quad P^* < P_1 < P_0, \quad P_0 < P_2 < \overline{P}_1.$$

$$B.\ \begin{cases} E(x, y; x', y'; \tilde{x}, \tilde{y}') > 0 \text{ on } \mathfrak{E} \text{ for } s_1 \leqq s \leqq s_0,\ \tilde{\theta} \neq \theta, \\ \qquad\qquad \text{except possibly for } s = s_0,\ \tilde{\theta} = \theta_0; \\ E(x, y; x', y'; \tilde{x}, \tilde{y}') > 0 \text{ on } \mathfrak{E} \text{ for } s_0 \leqq s \leqq s_2,\ \tilde{\theta} \neq \theta, \\ \qquad\qquad \text{except possibly for } s = s_0,\ \tilde{\theta} = \theta_0. \end{cases}$$

On the other hand, we have found above the following necessary conditions:

$$\Omega_0 \neq 0, \qquad P^* < P_1 \leqq P_0, \qquad P_0 < P_2 \leqq \overline{P}_1.$$

* *Dissertation*, pp. 31 and 32, where no explicit conditions are stated, but the implication is made that $\overline{P}_1$ need not always be the bound for minimizing extremals.

† *Vorlesungen*, § 50.

It is evident that there is *accord for* $\Omega_0 < 0$, *but contradiction for* $\Omega_0 > 0$.

We investigate now the behavior of the E-function in the neighborhood of P_0. We know that

$$E(x, y; x', y'; \bar{x}', \bar{y}') = F(x, y, \bar{x}', \bar{y}') - \bar{x}' F_{x'}(x, y, x', y') - \bar{y}' F_{y'}(x, y, x', y').*$$

Writing

$$E[x(s), y(s); x'(s), y'(s); \cos\bar{\theta}, \sin\bar{\theta}] \equiv E(s; \bar{\theta}),$$

$$E[\bar{x}(s), \bar{y}(s); \bar{x}'(s), \bar{y}'(s); \cos\bar{\theta}, \sin\bar{\theta}] \equiv \bar{E}(s; \bar{\theta}),$$

$$\phi[x(s), y(s), \bar{x}', \bar{y}'] \equiv \bar{\phi}(s),$$

$$\phi[\bar{x}(s), \bar{y}(s), \bar{x}', \bar{y}'] \equiv \bar{\bar{\phi}}(s),$$

we derive the equations

$$E_s(s; \bar{\theta}) = x'\bar{F}_x(s) + y'\bar{F}_y(s) - \bar{x}'\frac{d}{ds}F_{x'} - \bar{y}'\frac{d}{ds}F_{y'},$$

$$\bar{E}_s(s; \bar{\theta}) = \bar{x}'\bar{\bar{F}}_x(s) + \bar{y}'\bar{\bar{F}}_y(s) - \bar{x}'\frac{d}{ds}\bar{F}_{x'} - \bar{y}'\frac{d}{ds}\bar{F}_{y'}.$$

But $x(s), y(s)$ and $\bar{x}(s), \bar{y}(s)$ representing extremals, we have

$$\frac{d}{ds}F_{x'} = F_x, \quad \frac{d}{ds}F_{y'} = F_y, \quad \frac{d}{ds}\bar{F}_{x'} = \bar{F}_x, \quad \frac{d}{ds}\bar{F}_{y'} = \bar{F}_y.$$

Hence,

$$E_s(s; \bar{\theta}) = x'\bar{F}_x(s) + y'\bar{F}_y(s) - \bar{x}'F_x(s) - \bar{y}'F_y(s),$$

$$\bar{E}_s(s; \bar{\theta}) = \bar{x}'\bar{\bar{F}}_x(s) + \bar{y}'\bar{\bar{F}}_y(s) - \bar{x}'\bar{F}_x(s) - \bar{y}'\bar{F}_y(s),$$

and therefore

$$E_s(s_0; \theta_0) = \Omega_0, \qquad \bar{E}(s_0; \theta_0) = -\Omega_0. \dagger \tag{52}$$

Further, CARATHEODORY ‡ has shown that the Erdmann-Weierstrass corner conditions are equivalent to

$$\begin{cases} E(s_0; \theta_0) = 0, \\ \dfrac{\partial}{\partial\bar{\theta}} E(s_0; \bar{\theta}_0) = 0, \end{cases} \quad \text{or} \quad \begin{cases} \bar{E}(s_0; \theta_0) = 0, \\ \dfrac{\partial}{\partial\bar{\theta}} \bar{E}(s_0; \theta_0) = 0. \end{cases} \tag{53}$$

We expand now $E(s; \bar{\theta})$ by Taylor's expansion at $s = s_0$, $\bar{\theta} = \theta_0$, and find

$$\begin{aligned} E(s; \bar{\theta}) &= E(s_0; \bar{\theta}_0) + \frac{s - s_0}{1}\Omega_0 + \frac{\bar{\theta} - \theta_0}{1} E_{\bar{\theta}}(s_0; \theta_0) + \cdots, \\ \bar{E}(s; \bar{\theta}) &= \bar{E}(s_0; \theta_0) - \frac{s - s_0}{1}\Omega_0 + \frac{\bar{\theta} - \theta_0}{1} E_{\bar{\theta}}(s_0; \theta_0) + \cdots. \end{aligned} \tag{54}$$

* BOLZA, loc. cit., p. 243.

† This is the second form of Ω_0, referred to in the first footnote on p. 482.

‡ *Dissertation*, p. 8.

It follows that *if the corner conditions are fulfilled and* $\Omega_0 > 0$ *at the point* $s = s_0$, *then* $E(s; \bar{\theta})$ *takes the sign of* $s - s_0$, *while* $\overline{E}(s; \bar{\theta})$ *takes the sign of* $s_0 - s$, *for small values of* $|s - s_0|$.

Consequently, if $\Omega_0 > 0$, we have $E < 0$ in a vicinity of P_0 on both P_1P_0 and P_0P_2. This shows that *the sufficient conditions AIa and AIb are incompatible with B.* This *removes the contradiction referred to above.*

By means of (54) we can give furthermore a simple proof of a theorem previously proved by CARATHEODORY *:

"*If one follows a strong extremal* $\mathfrak{E}$ *up to a point* P_0 *where it ceases to be strong, and if at* P_0 *the invariant* Ω_0 *does not vanish (for some value* $\theta = \bar{\theta}_0$),† *then there is another extremal* $\overline{\mathfrak{E}}$ *passing through* P_0, *which begins to be strong at* P_0 *and which forms with* $\mathfrak{E}$ *a discontinuous solution of the problem.*"

If $\mathfrak{E}$ ceases to be strong at $P_0(s_0)$, then we must have the two relations

$$(55)\qquad \begin{aligned} E(s; \tilde{\theta}) &> 0 \qquad \text{for} \qquad s < s_0 \qquad (0 \leqq \tilde{\theta} \leqq 2\pi,\ \theta \neq \tilde{\theta}), \\ E(s; \tilde{\theta}) &< 0 \qquad \text{for} \qquad s > s_0, \end{aligned}$$

for at least one value of $\tilde{\theta}$, say $\bar{\theta}_0$, different from θ. From the continuity of E follows then

$$E(s_0; \bar{\theta}_0) = 0.$$

Hence by (54),

$$E(s; \tilde{\theta}) = \frac{s - s_0}{1}\Omega_0 + \frac{\tilde{\theta} - \theta_0}{1}E_{\tilde{\theta}}(s_0; \bar{\theta}_0) + \cdots.$$

Supposing for the moment $\Omega_0 < 0$, we see that if (55) is to be fulfilled, E must be of constant sign whenever $s - s_0$ keeps its sign.‡ But putting $\tilde{\theta} - \theta_0 = \lambda(s - s_0)$, we find

$$E(s; \tilde{\theta}) = (s - s_0)[\Omega_0 + \lambda E_{\tilde{\theta}}(s_0; \tilde{\theta}_0)] + \cdots,$$

from which it is evident that after the sign of $s - s_0$ is once fixed, E can be made positive as well as negative by a proper choice of λ, unless $E_{\tilde{\theta}}(s_0; \bar{\theta}_0) = 0$. Consequently, if all the hypotheses of the theorem are fulfilled, we may conclude that

$$E(s_0; \bar{\theta}_0) = 0, \qquad \frac{\partial}{\partial \bar{\theta}} E(s_0; \bar{\theta}_0) = 0.$$

This shows that the corner condition must be fulfilled at the point P_0 by the direction $\bar{\theta}_0$. By repeating with respect to $\overline{E}$ the above argument the second part of Caratheodory's theorem may easily be proved.

We shall defer to a later paper an example showing the application of the results of the last two sections.

THE UNIVERSITY OF CHICAGO,
June, 1908.

* Mathematische Annalen, vol. 62 (1906), p. 473.

† The statement in parenthesis is mine.

‡ This discussion is valid only for a neighborhood of P_0, a limitation which does not interfere however with our argument.

SETS OF METRICAL HYPOTHESES FOR GEOMETRY*

BY

ROBERT L. MOORE

In this paper is given a set of assumptions, C, concerning point, order † and congruence, which, together with a certain set of order assumptions O,‡ a continuity assumption K,‡ and a very weak parallel assumption P_0,‡ are sufficient for the establishment of ordinary Euclidean geometry. Each assumption of this set is independent of all the remaining ones. Moreover congruence is here an independent symbol in the sense that it would be impossible § to prove that if the order assumptions O, K and P_0 are true, then there must exist a relation satisfying the congruence assumptions C with reference to the points and order in terms of which O, K and P_0 are stated. According to VEBLEN ‖ congruence would not, however, be thus independent if P_0 were replaced even by his still comparatively weak parallel assumption XI.¶

If in this categorical set of assumptions there is substituted for K the assumption that *every segment has a middle point*,** and P_0 is replaced by the somewhat stronger assumption P_2,†† then follows ‡‡ a geometry §§ in which a theory of proportion holds and rigid motion is possible.

Other alternative sets of assumptions are discussed. Among these an interesting set is that obtained by substituting for P_0 the postulate P_2 and for K a postulate I_2 which may be roughly stated as follows: "*If B is within, and B' without, any circle α, then each semicircle with $B'B$ as diameter must have a point in common with the circle α.*"‖‖ Each assumption in the set composed of P_2, I_2

* Presented to the Chicago section of the American Mathematical Society, in a somewhat different form, April 22, 1905. Received for publication December 26, 1907.

† It would be impossible (see Theorem 1 of § 12) to formulate two sets of assumptions, one set, C, being stated in terms of point and congruence alone, and the other, O, being stated in terms of point and order alone, such that from O and C would follow a geometry, Euclidean with respect to the undefined symbols in terms of which O and C are stated.

‡ O, K and P_0 are all stated in terms of point and order.

§ Cf. § 10.

‖ These Transactions, vol. 5 (1904), pp. 343-384.

¶ Loc. cit. p. 346.

** Cf. end of § 1 and first part of § 6.

†† See § 6.

‡‡ Cf. Theorem 1 of § 6.

§§ A geometry which is a consequence of this set of assumptions is a consequence of HILBERT's Axiom-groups I–IV and conversely.

‖‖ Cf. § 7.

and the assumptions C and O is independent of the remaining ones, and moreover this set is sufficient for the establishment of a geometry in which not only is there a theory of proportion and rigid motion but also all ordinary rule and compasses constructions are possible.

In this paper "proof" is used to mean an *indication* of a demonstration. "Hence," "by," "therefore," etc., are intended as suggesting certain relations and not necessarily as describing exactly the logical dependence of one statement upon another.

I wish to thank Professor E. H. MOORE and Professor O. VEBLEN for suggestions and criticisms. Professor VEBLEN, who suggested the undertaking of this investigation, has not only made numerous suggestions and criticisms, but has given me much help in the way of actual collaboration.

Assumptions for Euclidean geometry.

O. VEBLEN's order "axioms" I and III-X.*

C. Assumptions concerning point and congruence, sometimes with reference to order, as follows:

C_{1_a}. *If B† is different from C and A' is different from B' then there exists a point C' such that $A'B'C'$‡ and $BC \equiv B'C'$.*§

C_{1_b}. *If B is different from C and A' is different from B' then there is not more than one point C' such that $A'B'C'$ and $BC \equiv B'C'$.*

C_2. *If A is different from B, A' is different from B', A'' is different from B'', $AB \equiv A'B'$ and $A'B' \equiv A''B''$, then $AB \equiv A''B''$.*

C_3. *If ABC, $A'B'C'$, $AB \equiv A'B'$ and $BC \equiv B'C'$, then $AC \equiv A'C'$.*

C_4. *If A, B and C are three non-collinear*‖ *points and A', B' and C' are three non-collinear points and CAD, $C'A'D'$, $AB \equiv A'B'$, $BC \equiv B'C'$, $CA \equiv C'A'$, $CD \equiv C'D'$, then $BD \equiv B'D'$.*¶

K. For the continuity assumption K may be taken either VEBLEN's XI or a DEDEKIND cut assumption stated for the points of a single segment as follows: *If there exists any segment then there exists some segment AB such that if it is*

* Loc. cit., pp. 344-346. From assumption C_{1_a} below it follows that if A and B are two different points there is a point C different from B such that ABC. VEBLEN's Axiom II may be seen to be a consequence of this proposition and his Axioms I, III-VIII (cf. a paper presented by me to the American Mathematical Society on October 26, 1907, but not yet published). Thus Axiom II is redundant in the set composed of VEBLEN's Axioms I-X together with my Axioms C.

† Capital letters are used to designate points.

‡ ABC used as a sentence means A, B and C are in the order ABC.

§ BC as a simple word, unmodified by "line" or "ray," means the segment BC, i. e., the set of all points X such that BXC.

‖ Cf. VEBLEN, loc. cit., p. 345. Instead of "three non-collinear points" one might, in stating this assumption, use the phrase "three points such that neither ABC, BCA, nor CAB."

¶ Cf. VERONESE, *Grundzüge der Geometrie*, pp. 254-260. C_4 and C_3 might be stated together in one assumption.

composed of two sets of points $[M]$* *and* $[N]$, *each set containing at least two points and no point* X *of either set being either the same as* Y_1 *or such that* Y_1XY_2 *where* Y_1 *and* Y_2 *are points of the other set, then there exists a point* C *such that* MCN *for every* M *and* N *different from* C.

With the help of O, from either of these assumptions can be deduced a theorem to the effect that this DEDEKIND cut proposition must apply to every segment as well as to the segment AB (cf. VEBLEN, loc. cit., pages 368 and 369).

P_0. *If there exists any straight line and a point not on it then there exists some straight line* a, *and some point* A, *not on it, such that if* a *and* A *lie in a plane*† β *then in the plane* β *there is not more than one straight line passing through* A *and having no point in common with* a.

§ 1. *Consequences of* O *and* C.

On the basis of O and C we have the following propositions:

Theorem 1. If A' is a point and AB is a segment, then on each ray ‡ starting from A' there is one and only one point B' such that $AB \equiv A'B'$.

Proof. See C_{1_a} and C_{1_b}.

Theorem 2. If ABC and C' is on ray $A'B'$ and $AB \equiv A'B'$ and $AC \equiv A'C'$, then $A'B'C'$.

Proof. There is, according to C_{1_a}, a point C'' such that $A'B'C''$ and $BC \equiv B'C''$. Hence, by C_3, $AC \equiv A'C''$. But, according to hypothesis, $AC \equiv A'C'$. Therefore, by C_{1_b}, C'' is C'. But $A'B'C''$.

Corollary. If B is on ray AC and B' is on ray $A'C''$ and $AB \equiv A'B'$ and $AC \equiv A'C'$, then $BC \equiv B'C'$.

Theorem 3.§ If A is distinct from B, then $AB \equiv AB$.

Proof. According to Theorem 1 there exists, on ray AB, a point B' such that $AB \equiv AB'$. Again there exists, on ray AB, a point B'' such that $AB' \equiv AB''$. But, then, according to C_2, $AB \equiv AB''$ and thus (see Theorem 1) B' is B''. Hence $AB' \equiv AB'$. Therefore $AB \equiv AB'$ and also $AB' \equiv AB'$. But if B is not B' then either ABB' or $AB'B$ and thus, by Theorem 2, $AB'B'$, which is impossible. So B is B'. But $AB \equiv AB'$. Hence $AB \equiv AB$.

Theorem 4. If $AB \equiv A'B'$, then $A'B' \equiv AB$.

Proof. According to Theorem 1 there exists, on ray AB, a point B'' such that $A'B' \equiv AB''$. So, by C_2, Theorem 3 and Theorem 1, B'' must be B. But $A'B' \equiv AB''$. Thus $A'B' \equiv AB$.

* The notation $[M]$ is used to denote a set of elements any one of which may be called M.

† Cf. VEBLEN, loc. cit., p. 345.

‡ If A and B are two distinct points, then the set of all points $[C]$ such that either C is B, or ACB, or ABC, is called a ray or half line starting from A. Such a ray is called the ray AB.

§ Cf. HILBERT's IV, 1 (see page 12 of HILBERT's *The Foundations of Geometry*, TOWNSEND's translation).

Theorem 5.* If $AB \equiv A'B'$ and $AB \equiv A''B''$, then $A'B' \equiv A''B''$.

Proof. This theorem is a logical consequence of Theorem 4 and C_2.

Theorem 6. If ABC and $AC \equiv A'C'$, then there exists one and only one point B' such that $A'B'C'$ and $AB \equiv A'B'$ and $BC \equiv B'C'$.

Proof. According to Theorem 1 there is one and only one point B' on ray $A'C'$ such that $AB \equiv A'B'$. According to Theorem 2 and its corollary, $A'B'C'$ and $BC \equiv B'C'$.

Definition 1. $A'B' < AB$ means: there exists a point P such that APB and $A'B' \equiv AP$. $A'B' > AB$ means $AB < A'B'$.

Theorem 7. If $AB > A'B'$, $AB \equiv CD$ and $A'B' \equiv C'D'$, then $CD > C'D'$.

Proof. According to definition 1 there exists a point P such that APB and $A'B' \equiv AP$. So, according to Theorem 6, there exists P' such that $CP'D$ and $AP \equiv CP'$. By C_2 it follows that $A'B' \equiv CP'$. Therefore, by hypothesis and Theorem 5, $C'D' \equiv CP'$. So $CD > C'D'$, according to Definition 1.

Theorem 8. If A is different from B and A' is different from B', then, of the three statements $AB \equiv A'B'$, $AB > A'B'$, $AB < A'B'$, one and only one is true.

Theorem 9. If $AB > A'B'$ and $A'B' > A''B''$, then $AB > A''B''$.

Definition 2. $AB + CD$ means the segment AE, where E is a point such that ABE and $CD \equiv BE$. If $AB > CD$ then $AB - CD$ means the segment AE where E is a point such that BEA and $CD \equiv BE$.

Theorem 10. If $AB \equiv A'B'$, $CD \equiv C'D'$, then $AB + CD \equiv A'B' + C'D'$. We have the following rules of combination: $AB + CD \equiv CD + AB$, $(AB + CD) + EF \equiv AB + (CD + EF)$, $AB - CD \equiv EF$ if and only if $AB \equiv CD + EF$.

A relation (called congruence) between angles† is introduced by the following definition:

Definition 3.‡ $\measuredangle ABC \equiv \measuredangle A'B'C'$ means that there exist points D, D', E, E', on rays BA, $B'A'$, BC, $B'C'$ respectively, such that $BD \equiv B'D'$, $DE \equiv D'E'$, $EB \equiv E'B'$.

Theorem 11. If $\measuredangle BAC \equiv \measuredangle B'A'C'$, $BA \equiv B'A'$, $CA \equiv C'A'$, then $BC \equiv B'C'$.

Proof. By hypothesis and Definition 3 there exist on rays AB, AC, $A'B'$, $A'C'$, points D, E, D', E' respectively such that $AD \equiv A'D'$, $DE \equiv D'E'$, $EA \equiv E'A'$. By Theorem 1 there exist points F, G, F', G', such that BAF, CAG, $B'A'F'$, $C'A'G'$, $AF \equiv A'F'$, $AG \equiv A'G'$. By hypotheses and C_4 it follows that $EF \equiv E'F'$ and therefore, by Theorem 2 and another application of C_4, it may be proved that $FG \equiv F'G'$. Treating A, B, C, A', B', C',

* See HILBERT's IV, 2, loc. cit., p. 12.

† An angle is a point together with two non-collinear rays which start from that point.

‡ Cf. VERONESE, loc. cit., p. 257.

with reference to A, G, F, A', G', F' as A, G, F, A', G', F' were treated with reference to A, D, E, A', D', E', one may prove that $BC \equiv B'C'$.

Theorem 12.* $\measuredangle ABC \equiv \measuredangle ABC$.

Proof. Use Definition 3 and Theorem 3.

Theorem 13. If $\measuredangle ABC \equiv \measuredangle A'B'C'$, then $\measuredangle A'B'C' \equiv \measuredangle ABC$.

Proof. Use Definition 3 and Theorem 4.

Theorem 14.† If $\measuredangle ABC \equiv \measuredangle A'B'C'$ and $\measuredangle ABC \equiv \measuredangle A''B''C''$, then $\measuredangle A'B'C' \equiv \measuredangle A''B''C''$.

Proof. Use Definition 3 and Theorems 11 and 5.

Definition 4. Two angles are supplementary to (supplements of) each other if and only if they have a common vertex and one common side and their other sides form two rays of the same straight line.

Two angles are vertical angles if and only if they have a common vertex and their four sides, together with this common vertex, make up two straight lines.

Theorem 15. If $\measuredangle HAK \equiv \measuredangle H'A'K'$ then each angle that is supplementary to $\measuredangle HAK \equiv$ each angle that is supplementary to $\measuredangle H'A'K'$, and the angle that is vertical to $\measuredangle HAK \equiv$ the angle that is vertical to $\measuredangle H'A'K'$.

Proof. See Theorem 11 and proof thereof.

Corollary. Vertical angles are congruent to each other.

Definition 5. $\triangle ABC \equiv \triangle A'B'C'$ means: $AB \equiv A'B'$, $BC \equiv B'C'$, $CA \equiv C'A'$, $\measuredangle ABC \equiv \measuredangle A'B'C'$, $\measuredangle BCA \equiv \measuredangle B'C'A'$, $\measuredangle CBA \equiv \measuredangle C'B'A'$.‡

Theorem 16. $\triangle ABC \equiv \triangle A'B'C'$ if $AB \equiv A'B'$, $BC \equiv B'C'$, $CA \equiv C'A'$.

Proof. See Definitions 3 and 5.

Theorem 17.§ $\triangle ABC \equiv \triangle A'B'C'$ if $AB \equiv A'B'$, $BC \equiv B'C'$, and $\measuredangle ABC \equiv \measuredangle A'B'C'$.

Proof. See Theorem 11 and Definitions 3 and 5.

Theorem 18. If $\measuredangle ABC \equiv \measuredangle A'B'C'$ and D is in $\| \measuredangle ABC$, then there exists a point D' in $\measuredangle A'B'C'$ such that $\measuredangle ABD \equiv \measuredangle A'B'D'$ and $\measuredangle DBC \equiv \measuredangle D'B'C'$.

Definition 6. $\measuredangle B'A'C' < \measuredangle BAC$ means: there is a point, P, in $\measuredangle BAC$ such that $\measuredangle B'A'C' \equiv \measuredangle BAP$. $\measuredangle B'A'C' > \measuredangle BAC$ means: $\measuredangle BAC < \measuredangle B'A'C'$.

Theorem 19. If $\measuredangle BAC > \measuredangle B'A'C'$, $\measuredangle BAC \equiv \measuredangle DEF$, $\measuredangle B'A'C' \equiv \measuredangle D'E'F'$, then $\measuredangle DEF > \measuredangle D'E'F'$.

Theorem 20. If $\measuredangle BAC > \measuredangle B'A'C'$ and $\measuredangle B'A'C' > \measuredangle B''A''C''$, then $\measuredangle BAC > \measuredangle B''A''C''$.

Theorem 21. If ABC is a $\triangle$ and $AC \equiv BC$, then $\measuredangle ABC \equiv \measuredangle ACB$.

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* See HILBERT's IV, 4, loc. cit., page 14.

† See HILBERT's IV, 5, loc. cit., page 14. $\triangle ABC$ (triangle ABC) means the three non-collinear points A, B, and C.

‡ See VERONESE, loc. cit., page 254.

§ See HILBERT's, IV, 6, loc. cit., page 15.

‖ If ABC is an angle, the two rays, BA and BC, together with the vertex B, divide the plane ABC into two regions. That one of these regions which contains points P such that APC is called the interior of $\measuredangle ABC$. Cf. VEBLEN, loc. cit., pp. 363–365.

Proof. Use Theorem 3 and Definition 3.

Definition 7. "A middle point of the segment AB" means a point M, on the straight line AB, such that $AM \equiv MB$.

Theorem 22. If M is a middle point of AB, then AMB.

Proof. Use Definition 7 and C_{1_b}.

Theorem 23. No segment has two middle points.

Proof. Use Definition 1 and Theorems 22, 7 and 8.

I do not know as yet whether from O and C it follows that every segment has one middle point. This proposition (statement) that *every segment has a middle point* will be denoted by the symbol M.

§ 2. *Two consequences of O, C and M.*

In § 1 a number of propositions were deduced as consequences of O and C. In the present paragraph it will be shown that, if M is assumed in addition to O and C, then follow two propositions concerning angles.

The symbol A_1 will be used to denote the proposition: *If ABC, $AB \equiv BC$, and D is not on the straight line AB, then there is, in the half-plane $D - AB$,* a point E such that $AD \equiv BE$ and $BD \equiv CE$.*†

The symbol A_{-2} denotes the proposition: *If C is not on the straight line AB, then, in the half-plane $C - AB$, there is no point C', different from C, such that $AC \equiv AC'$ and $BC \equiv BC'$.*‡

Theorem 1. From O, C and M follows A_{-2}.

Proof. If this theorem is not true, then there exists a space for which O, C and M are true but which contains four distinct coplanar points A, B, C', C'' such that C' and C'' are on the same side of AB and moreover $BC' \equiv BC''$ and $C'A \equiv C''A$. In case the straight lines $C'C''$ and AB have a point P in common, then with use of Theorems 15, 16, 17 and 3 it may be seen that $\triangle APC' \equiv \triangle APC''$, and thus $PC' \equiv PC''$, but this is impossible according to C_{1_b} and the hypothesis that C' and C'' are on the same side of AB. In case the straight lines $C'C''$ and AB have no point in common, then A and B are on the same side of the straight line $C'C''$. According to M and Theorem 22, there is a point S such that $C''SC'$ and $C''S \equiv SC'$. There exists, as may be seen by use of O, a point P such that PSA and segment PB has a point S' in common with segment $C'C''$. Consider $\triangle$'s ASC' and ASC''. $AS \equiv AS$ by Theorem 3, while, by hypothesis, $AC' \equiv AC''$ and $C'S \equiv SC''$. Hence, by Theorem 16, $\triangle ASC' \equiv \triangle ASC''$. With use of C_2, C_4 and The-

* By a half-plane is meant one of the two regions into which a plane is decomposed by a straight line which lies in it (cf. VEBLEN, loc. cit., pp. 363–365). If the straight line AB thus decomposes a plane into two regions and D lies in one of these regions then this region is called the half-plane $D - AB$.

† Thus, according to Definition 3, $\measuredangle BAD \equiv \measuredangle CBE$ and A_1 is, then, a weak form of the proposition given in the statement of Theorem 4 of §5. Cf. HILBERT's IV, 4, loc. cit., p. 14.

‡ Cf. HILBERT's IV, 4, loc. cit., p. 14.

orem 3 it follows that $PC'' \equiv PC'$. Hence, by hypothesis, Theorem 3 and Definition 3, it follows that $\sphericalangle C''BS' \equiv \sphericalangle C'BS'$ and, consequently, by means of Theorems 17 and 3 and Definition 5, that $C''S' \equiv S'C'$, but this is impossible according to theorem 23.

*Corollary.** Given an angle BAC and a ray $A'B'$, there is, in a given plane containing $A'B'$ and on a given side of $A'B'$, not more than one ray $A'C'$ such that $\sphericalangle BAC \equiv \sphericalangle B'A'C'$.

Theorem 2. From O, C and M follows A_1.

Proof. Suppose ABC, $AB \equiv BC$ and D is not on the straight line AC. According to M and Theorem 22, there is a point S such that ASB and $AS \equiv SB$. By C_{1_a} there exists a point E' such that DSE' and $DS \equiv SE'$. According to Theorem 17 and Corollary to theorem 15, $\sphericalangle SAD \equiv \sphericalangle SBE'$. But by C_{1_a} there is a point E such that $E'BE$ and $AD \equiv BE$. By theorems 13 and 14 and Corollary to Theorem 15, $\sphericalangle DAS \equiv \sphericalangle CBE$. But $AB \equiv BC$ and $AD \equiv BE$. Hence, by theorem 11, $BD \equiv CE$.

§ 3. *Consequences of O, C and A_{-2}.*

In § 2 it has been shown that if M is assumed in addition to O and C, then there follow the two propositions A_1 and A_{-2}. On the basis of O, C and A_{-2} we have the following propositions.

Theorem 1. $\triangle ABC \equiv \triangle A'B'C'$ if $\sphericalangle ABC \equiv \sphericalangle A'B'C'$, $\sphericalangle BAC \equiv \sphericalangle B'A'C'$ and $AB \equiv A'B'$.

Proof. According to Theorem 1 of § 1 there is a point C'' on ray $A'C'$ such that $AC \equiv A'C''$. According to hypothesis and Theorem 17, $\triangle ABC \equiv \triangle A'B'C''$ and thus (see Definition 5) $\sphericalangle ABC \equiv \sphericalangle A'B'C'$. But $\sphericalangle ABC = \sphericalangle A'B'C'$. By A_{-2} and Theorem 1 of § 1 it follows that either C'' is C' or, if not, then B' is on the straight line $C''C'$ and, thus, on the straight line $A'C'$, contrary to hypothesis. So C'' is C'. But $\triangle ABC \equiv \triangle A'B'C''$. Hence $\triangle ABC \equiv A'B'C'$.

Theorem 2.† Of the three statements, $\sphericalangle BAC < \sphericalangle B'A'C$, $\sphericalangle BAC \equiv \sphericalangle B'A'C'$, $\sphericalangle BAC > \sphericalangle B'A'C'$, not more than one is true.

Proof. Make use of Definition 5 and A_{-2}. For instance suppose $\sphericalangle BAC > \sphericalangle B'A'C'$. Then, according to Definition 6, there exists a point D, in $\sphericalangle BAC$, such that $\sphericalangle B'A'C' \equiv \sphericalangle BAD$. So, if $\sphericalangle BAC \equiv \sphericalangle B'A'C'$, then $\sphericalangle BAC \equiv \sphericalangle BAD$, and this could not be, in view of A_{-2}; while, if $\sphericalangle BAC < \sphericalangle B'A'C'$, then there exists, in $\sphericalangle B'A'C'$, a point D' such that $\sphericalangle BAC \equiv \sphericalangle B'A'D'$. Then, according to original hypothesis and Theorem 19, $\sphericalangle B'A'D' > \sphericalangle B'A'C'$. Hence there exists, in $\sphericalangle B'A'D'$, and at the same

*See HILBERT's IV, 4, loc. cit., p. 14.

†Observe that it is not here stated that *one* of these statements *is* true. See Corollary to Theorem 2 of § 4.

time in $\measuredangle B'A'C'$, a point G'' such that $\measuredangle B'A'C' \equiv \measuredangle B'A'G''$, but this could not be according to A_{-2}. So, if $\measuredangle BAC > \measuredangle B'A'C'$, then neither $\measuredangle BAC < \measuredangle B'A'C'$ nor $\measuredangle BAC \equiv \measuredangle B'A'C'$.

Theorem 3. If HGF, and I is not on the straight line HF and A is different from C, and B, D, A and C are coplanar but neither B nor D is on the straight line AC, and $\measuredangle BAC \cong \measuredangle FGI$ and $\measuredangle ACD \cong \measuredangle IGH$, then the rays CD and AB do not meet.

Proof. Of course, the rays AB and CD cannot meet unless they are on the same side of the straight line AC. Suppose they are on the same side of AC. According to hypothesis and Definition 6, there exist points P and S such that either ray CS is ray CD or ray CS is in $\measuredangle DCA$ and moreover either ray AP is ray AB or ray AP is in $\measuredangle CAB$ and furthermore $\measuredangle ACS \equiv \measuredangle HGI$ and $\measuredangle CAP \equiv \measuredangle FGI$. Evidently, then, rays CD and AB cannot meet unless rays CS and AP meet. Suppose rays CS and AP do meet in a point M. There exists, according to C_{1_2}, a point M' such that MCM' and $AM \equiv CM'$. Now, by Theorems 15, 13 and 14, $\measuredangle CAM \equiv \measuredangle ACM'$. Hence, by Theorems 3 and 17 and Definition 5, $\measuredangle CAM' \equiv \measuredangle ACM$. But, by Theorems 15, 13 and 14, $\measuredangle ACM \equiv \measuredangle P'AC$, where P' is any point such that $P'AP$. It follows, by means of A_{-2}, that the points M, A and M' are collinear. So the straight lines AP and CS unite in two* points M and M', but this is impossible. Therefore rays AP and CS do not meet.

Corollary. If HGF, and I is not on the straight line HF, and A, B and C are three non-collinear points, and D is in the half-plane $B - AC$, and $\measuredangle BAC \equiv \measuredangle FGI$ and $\measuredangle ACD \equiv \measuredangle IGH$, then the lines CD and AB have no point in common.

Theorem 4. If ABC is a triangle and $BC > AC$, then $\measuredangle A > \measuredangle B$.

Proof. By hypotheses, Theorem 7, and Definition 1, there exists, between B and C, a point D such that $AC \equiv CD$. Since D is between B and C, it follows from Theorem 12 and Definition 6 that $\measuredangle BAC > \measuredangle DAC$. But, by construction and Theorem 21, $\measuredangle DAC \equiv \measuredangle ADC$. Hence, by Theorems 12 and 19, $\measuredangle BAC > \measuredangle CDA$. But, by Theorem 2 and Corollary 2 to Theorem 3 of § 3, $\measuredangle CDA > \measuredangle ABC$. Hence, by Theorems 3 and 19 of § 1, $\measuredangle BAC > \measuredangle ABC$.

Theorem 5. If D is within $\measuredangle BAC$ and D' is within $\measuredangle B'A'C'$ and $\measuredangle BAD \equiv \measuredangle B'A'D'$ and $\measuredangle DAC \equiv \measuredangle D'A'C'$, then $\measuredangle BAC \equiv B'A'C'$.

Proof. It may be easily seen that there exist points E, F, E', G' on rays AC, AD, $A'C'$, $A'D'$ respectively such that $AE \equiv A'E'$, $AF \equiv A'F'$, and EF and $E'F'$ cut rays AB and $A'B'$ in two points G and G' respectively. By hypothesis and Theorem 17, $\measuredangle AFE \equiv \measuredangle A'F'E'$ and $EF \equiv \measuredangle E'F'$. But it is clear that EFG and $E'F'G'$. Hence, by Theorem 15,

* That M and M' are distinct is a consequence of O.

$\measuredangle AFG \equiv \measuredangle A'F'G'$. But $\measuredangle GAF \equiv \measuredangle G'A'F'$ and $AF \equiv A'F'$. Hence, by Theorem 1, $FG \equiv F'G'$. But $EF \equiv E'F'$. Hence, by C_3, $EG \equiv E'G'$. But $AE \equiv A'E'$ and $AG \equiv A'G'$. Hence, by Definition 3, $\measuredangle BAC \equiv \measuredangle B'A'C'$.

Definition 7. A right angle is an angle that is congruent to its supplement. A straight line a is perpendicular to a straight line b if and only if they intersect in a point O and there exist two points, A and B, different from O, A lying on a and B lying on b, such that $\measuredangle AOB$ is a right angle.

Corollary. Every angle which is congruent to a right angle is itself a right angle. If the straight line a is perpendicular to the straight line b, then b is perpendicular to a and if A and B are *any* two points different from O, A lying on a and B lying on b, then $\measuredangle AOB$ is a right angle.

Proof. See Theorems 12, 14 and 15 of § 1.

Theorem 6. In each plane there is a right angle.

Proof. Suppose A, B, C, are three distinct non-collinear points of a given plane. According to Theorem 1 of § 1 there exists, on ray AC, a point B' such that $AB \equiv AB'$. There is a point D such that ADB and, according to Theorem 6 of § 1, there is a point D' such that $B'D'A$, $BD \equiv B'D'$ and $DA \equiv D'A$. According to O, there is a point P such that BPD' and $B'PD$. Now $\measuredangle B'BA \equiv \measuredangle BB'A$ (see Theorem 21). Hence, by Theorems 3 and 17 of § 1, $\triangle BB'D \equiv \triangle B'BD'$. Thus $\measuredangle B'D'B \equiv \measuredangle B'DB$ and therefore, by Theorem 15, it follows that $\measuredangle PD'A \equiv \measuredangle PDA$. Moreover, by Theorems 17 and 12 and Definition 5, $\measuredangle PBD \equiv \measuredangle PB'D'$. Hence $\triangle B'PD' \equiv \triangle BPD$ according to Theorem 1. Thus $PD' \equiv PD$. Now also $DA \equiv D'A$. Hence, by Theorem 3 of § 1 and Definition 3, $\measuredangle B'AP \equiv \measuredangle BAP$. According to O, there is a point M, on AP, such that $B'MB$. By Theorems 17 and 3 and Definition 5, $MB' \equiv MB$ and $\measuredangle AMB' \equiv \measuredangle AMB$. Thus $\measuredangle AMB$ is a right angle (see Definition 7).

Corollary. If BAC is an angle, then there exists, within $\measuredangle BAC$, a point O such that $\measuredangle BAO \equiv \measuredangle OAC$.

Theorem 7. If there is one perpendicular p to a straight line a, then, through each point of a, and lying in the plane pa, there is one and only one straight line perpendicular to a.

Proof. Suppose O is the point at which p intersects a, C is any other point of a, and P is any other point of p. By C_{1_a} there exists a point P' such that POP' and $PO \equiv OP'$. There is also a point P'' such that $P'CP''$ and $PC \equiv CP''$. There is (see Corollary to Theorem 6) a ray CM in $\measuredangle PCP''$ such that $\measuredangle PCM \equiv \measuredangle MCP''$. Now, also, $\measuredangle PCO \equiv \measuredangle P'CO \equiv \measuredangle P''CO'$, where OCO'. Hence (see Theorem 5) $\measuredangle MCO \equiv \measuredangle MCO'$. Hence, by definition 7, MC is perpendicular to a at the point C.

Suppose there is, in the plane pa, another perpendicular to a at the point C. Suppose N is a point of this new perpendicular on the same side of a as M and

suppose MCM' and NCN'. Ray CN must lie either in $\measuredangle OCM$ or in $\measuredangle O'CM$. Suppose it lies in $\measuredangle OCM$. Then ray CN' lies in $\measuredangle O'CM'$. According to Definition 7, Theorem 12, and Theorem 15, $\measuredangle MCO \equiv \measuredangle M'CO$ and $\measuredangle NCO \equiv \measuredangle N'CO$. By Theorem 18, there is a point N'' in $\measuredangle M'CO$ such that $\measuredangle OCN \equiv OCN''$. But $\measuredangle OCN \equiv \measuredangle OCN'$ and ray CN' is in $\measuredangle O'CM'$ and thus is different from CN''. But this is impossible according to A_{-2}. Thus CN cannot be in $\measuredangle OCM$. Similarly it cannot be in $\measuredangle O'CM$. There is, then, in the plane pa and passing through C, not more than one straight line perpendicular to a.

Theorem 8. If there is one perpendicular p to a straight line a, and P is any point in the plane pa, then through P, and lying in this plane, there is one and only one perpendicular to a.

Proof. If P is on a cf. Theorem 7. If P is not on a, suppose F is a point of p on the same side of a as P and O is the point where p meets a. If P is on OF, then certainly, through P, there is a perpendicular (OF) to AB. If P is not on OF, it is on one side of it. Suppose H is a point of a that is on the same side of OF as P. Then P lies in $\measuredangle HOF$. There is (by C_{1_a}) a point F' such that FOF' and $OF \equiv OF'$. According to Theorem 18 there is, in $\measuredangle HOF'$, a point P' such that $\measuredangle POH \equiv \measuredangle HOP'$. On ray OP' there is a point P'' such that $OP \equiv OP''$. Since P and P'' are on opposite sides of OH and on the same side of OF, there is a point M of ray OH such that PMP''. According to Theorems 3 and 17 of § 1, $\triangle OMP \equiv \triangle OMP''$. So $\measuredangle PMO \equiv \measuredangle P''MO$ and thus PM is perpendicular to a.

Now suppose that through P there is, in this same plane, another perpendicular to a. Suppose this new perpendicular meets a in the point M'. There is a point p''' such that $PM'P'''$. According to hypothesis, Definition 7 and Theorems 13 and 14, $\measuredangle MM'P \equiv \measuredangle MM'P'''$ and $\measuredangle M'MP \equiv \measuredangle M'MP''$. According to Theorem 17, $\triangle PMM' \equiv \triangle P''MM'$. So $\measuredangle PM'M \equiv \measuredangle MM'P''$. But $\measuredangle PM'M \equiv \measuredangle P'''M'M$. Hence, according to A_{-2}, P'' must lie on ray $M'P'''$ and thus the straight lines MP and $M'P$ intersect in two points P and P'', but this is impossible. So there is through P, and in a given plane containing a, not more than one perpendicular to a.

Theorem 9. If in the plane of two intersecting straight lines there is a perpendicular to one of them, then there is in this plane a perpendicular to the other one.

Proof. Suppose that the straight lines a and b intersect in a point O and there is a perpendicular to a lying in the plane ab. Then, if B is any point of b other than O, it follows, by Theorem 8, that there is a point A on a such that BA is perpendicular to a. In case A is O, then a is perpendicular to b, and the conclusion of Theorem 9 is evidently verified. In case A is not O, then there exist points A' and B', on b and a respectively, such that $OA \equiv OA'$,

$OB \equiv OB'$. By Theorems 12 and 17, $\measuredangle BAO \equiv \measuredangle B'A'O$. But $\measuredangle BAO$ is a right angle. Hence, by the corollary to Definition 7, $\measuredangle B'A'O$ is a right angle. So BA is perpendicular to b.

Theorem 10. If a is any straight line and P is any point, then in a given plane α containing P and a there is, through P, one and only one straight line perpendicular to a.

Proof. By Theorem 6, there exists in α some straight line a' to which there is a perpendicular lying in α. In case a' is not a, suppose A and A' are two points of a and a' respectively. By hypothesis and Theorem 9 there is in α a perpendicular to AA', and therefore, by Theorem 9, there is in α a perpendicular to a. Hence, by Theorem 8, through each point of α there is, in α, one and only one perpendicular to a.

Theorem 11. If a and b are two intersecting straight lines then there exist in the plane ab straight lines a' and b' which are perpendicular to a and b respectively and are such that each angle that a' makes with $a \equiv$ each angle that b' makes with b.

Proof. See Theorem 10 and proof of Theorem 9.

§ 4. *Consequences of* O, C, A_{-2} *and* A_1.

In § 2 it was shown that A_1 as well as A_{-2} was a consequence of O, C and M. On the basis of O, C, A_1 and A_{-2} we have the following propositions:

Theorem 1. Any two coplanar right angles are congruent to each other.

Proof. If a side of one right angle is collinear with a side of another which is coplanar with it, it may be easily seen, with help of A_1, Definition 7 and Theorems 1, 15, 13 and 14, that they are congruent to each other. From this result and Theorem 11 of § 3, it follows that two coplanar right angles are congruent to each other if the straight line which contains a side of one intersects the straight line which contains a side of the other. From this, O, Theorem 14 of § 1, and Theorem 10 of § 3, it follows that any two coplanar right angles are congruent to each other.

Theorem 2*. If A, B, C are not collinear and D', A', B' are not collinear, but A, B, C, D', A', B' are coplanar, then, in the half-plane $D' - A'B'$, there is a point C' such that $\measuredangle BAC \equiv \measuredangle B'A'C'$.

Proof. According to Theorem 10 of § 3, there exist points, E in the half-plane $C - AB$, and E' in the half-plane $D' - A'B'$, such that $\measuredangle EAB$ and $\measuredangle E'A'B'$ are right angles. According to Theorem 1, $\measuredangle EAB \equiv \measuredangle E'A'B'$. Suppose ray AC lies within $\measuredangle EAB$. Then, according to Theorem 18 of § 1 and Theorem 1 of § 4, there exists, in $\measuredangle E'A'B'$, a ray $A'C'$, such that $\measuredangle BAC \equiv \measuredangle B'A'C'$. Proceed in a similar manner if AC is within $\measuredangle B_1AE$ (where B_1AB).

* Cf. HILBERT's, IV, 4, loc. cit., page 14.

Corollary. If ABC and $A'B'C'$ are two coplanar angles then either $\measuredangle ABC < \measuredangle A'B'C'$, $\measuredangle ABC \equiv \measuredangle A'B'C'$ or $\measuredangle ABC > \measuredangle A'B'C'$.*

Theorem 3. If ABC is a triangle and B is between A and D, then $\measuredangle DBC > \measuredangle CAB$.

Proof. By hypothesis and Theorem 3 of § 3, neither $\measuredangle DBC < \measuredangle CAB$ nor $\measuredangle DBC \equiv \measuredangle CAB$. Hence, by corollary to Theorem 2 of § 4, $\measuredangle DBC > \measuredangle CAB$.

Theorem 4. If ABC is a triangle and $BC > AC$, then $\measuredangle BAC > \measuredangle ABC$ and, conversely, if $\measuredangle BAC > \measuredangle ABC$, then $BC > AC$.

Proof. I. Suppose $BC > AC$. Then, by Definition 1, there exists between B and C a point D such that $AC \equiv CD$. Since D is between B and C, it easily follows, from O and Definition 6, that $\measuredangle BAC > \measuredangle DAC$. But, by construction and Theorem 21 of § 1, $\measuredangle DAC \equiv \measuredangle ADC$. Hence, by Theorems 12 and 19 of § 1, $\measuredangle BAC > \measuredangle CDA$. But, by Theorem 3 of § 4, $\measuredangle CDA > \measuredangle ABC$. Hence by Theorem 20 of § 1, $\measuredangle BAC > \measuredangle ABC$.

2. Conversely, suppose $\measuredangle BAC > \measuredangle ABC$. By Theorem 8 of § 1 either $BC < AC$, $BC \equiv AC$ or $BC > AC$. If $BC < AC$ or $BC \equiv AC$ then, by Theorem 21 of § 1 and first part of present theorem, either $\measuredangle BAC < \measuredangle ABC$ or $\measuredangle BAC \equiv \measuredangle ABC$; and each of these, by Theorem 2 of § 3, is contrary to hypothesis. Hence $BC > AC$.

Theorem 5. If $\measuredangle ABC$ is a right angle, or $\measuredangle ABC >$ a right angle, of the plane ABC, then $AC > AB$.

Proof. From hypotheses, Definitions 6 and 7, Theorems 19, 20 and 12 of § 1, Theorems 3, 2 and 10 of § 3 and Theorem 1 of § 4, it follows that neither $\measuredangle ACB \equiv \measuredangle ABC$ nor $\measuredangle ACB > \measuredangle ABC$. Hence, by the corollary to Theorem 2 of § 4, $\measuredangle ACB < \measuredangle ABC$. Hence, by Theorem 4 of § 4 and Definition 6, $AC > AB$.

Theorem 6. If ABC is a triangle, then $AB + AC > BC$.

Proof. By C_{1_a}, there exists a point D such that CAD and $AB \equiv AD$. By Theorem 21 of § 1 $\measuredangle BDC \equiv \measuredangle DBA$. Hence, by O and Definition 6, $\measuredangle DBC > \measuredangle CDB$. Hence, by Theorem 4 of § 4, $DC > BC$. But, according to Definition 2, $DC = AB + BC$. Hence $AB + AC > BC$.

Corollary. If ABC is a triangle and $AB > AC$, then $AB - AC < BC$.

§ 5. *Euclidean and Bolyai-Lobachevskian geometry.*

It is desired to prove that from O, C, K and P_0 follows Euclidean geometry and from O, C, K and the denial of P_0 follows Bolyai-Lobachevskian geometry.

Definition 8. Two straight lines are said to be parallel to each other if they lie in the same plane and have no point in common.

Theorems 1–5 are based on the assumptions O, C, K.

* Cf. Theorem 2 of § 3.

Theorem 1. Every segment has a middle point.

Proof. Suppose AB is a segment. Certainly there exists a point X, of AB, such that $AX < XB$ and a point Y, of AB, such that $AY > YB$. For suppose P is any point of AB. By Theorem 8 of § 1, either $AB < PB$, $AP \equiv PB$ or $AP > PB$. If $AP > PB$, then, by Theorems 6 and 3 of § 1, there is a point P' of AB such that $AP \equiv BP'$ and $PB \equiv P'A$ and, by Theorem 7 of § 1 and Definition 1, it follows that $AP' < P'B$. Similarly, if $AP < PB$, there exists a point P' such that $AP' > PB$. Finally, if $AP \equiv PB$, any point D between A and P would be such that $AD < DP$, and there would exist a point D' such that $AD' > D'P$. It is true then that there is at least one point X of AB such that $AX < XB$ and at least one point Y of AB such that $AY > YB$. By Theorem 8 of § 1 Y could not be identical with X. The points of AB may then be divided into two classes, the class, $[X]$, of all points such that $AX < XB$ and the class, $[Y]$, of all other points of AB. By Definition 1, O, and Theorems 9 and 8 of § 1 it is clear that no X is between two Y's and no Y is between two X's. Hence, by K, there exists a point M such that XMY for every X and Y which are different from M. Suppose it were true that $AM < MB$. Then, by Theorems 6 and 3 of § 1, there would exist a point N of AB such that $AM \equiv BN$ and $MB \equiv NA$. Clearly M would be between A and N. Hence, by a previous argument, it would follow that there exists a point L, between M and N, such that $ML < LN$. But, from this together with the fact that $AM \equiv BN$, it would follow, from Definition 1 and Theorems 10 and 7 of § 1, that $AL < LB$. But this would be impossible. For then L would be an X and we would thus have an X such that AMX. Similarly it would be impossible that $AM > MB$. Hence $AM \equiv MB$.

This theorem having been established, it is clear that every proposition, given in the preceding part of this paper as a consequence of O, C and M, is also a consequence of O, C and K. Such propositions will therefore be freely referred to in this section. It is to be remembered that A_1 and A_{-2} are propositions of this kind.

Definition 9. If O is a point of a plane β and s is a segment, then the set of all points $[P]$ of the plane β, such that $OP \equiv s$ is called a circle.

Theorem 2. Every circle is a Jordan curve.

Proof. In his article "Theory of plane curves in non-metrical analysis situs,"* VEBLEN has given three sets of conditions as sufficient in order that a simple closed set of points, existing in a space in which K and the plane axioms of the set O hold true, should be a Jordan curve. These conditions are called the conditions of linear order, ordinal continuity, and geometrical continuity respectively.

* These Transactions, vol. 6 (1905), pp. 83–98.

If O is the center of a circle α, then every ray starting from O contains one and only one point of α.

The points A, B, C, D of α are said to be in the order $ABCD$ if the rays OA, OB, OC, OD are in the order $OA-OB-OC-OD$. From O, K and this correspondence, it follows that the points of α satisfy, with reference to this order, the conditions of linear order and ordinal continuity.

The condition of geometrical continuity will evidently be satisfied if the proposition can be established that, given any point P of α and any triangle Δ_P which contains P, then there exist in that triangle two points, P_1 and P_2 of α, such that every point on the arc P_1PP_2 will lie within Δ_P. To establish this proposition, proceed as follows: Given that Δ_P contains the point P of the circle α, consider a segment s which is shorter than each of the perpendiculars from O to the sides of Δ_P. These perpendiculars and this segment exist by Theorem 10 of § 3, Theorems 7, 8 and 9 of § 1, Definition 1 and O. By hypothesis, O, Theorem 5 of §4 and Definition 1 and Theorem 9 of § 1, it follows that every point P', such that $PP' < s$, must be within Δ_P. By Theorem 10 of § 3, there exists, in the plane of α, a straight line a perpendicular to OP at the point P. On a there exist, by Theorem 1 of § 1, two points E and F, one on each side of P, such that $s \equiv PE$ and $s \equiv PF$. Each of the rays OE and OF contains a point of α. Call these points P_1 and P_2 respectively. Let X be any point of α on the arc P_1PP_2. Then either X is P or it is in $\measuredangle POE$ or $\measuredangle POF$. Suppose X is in $\measuredangle POE$. Then the ray OX contains a point M such that PME. By Theorem 5 of § 4, $OM > OP$. Hence $OM > OX$ and therefore OXM. But from the fact that $OP \equiv OX$ and Theorem 21 of § 1 it follows that $\measuredangle OXP \equiv \measuredangle OPX$, and thus (see Theorem 3 of § 3) $\measuredangle OXP <$ a right angle (of the plane α). Hence clearly, $\measuredangle PXM >$ a right angle (of the plane α) and therefore, by Theorem 5 of § 4, it follows that $PX < PM$. But $PM < PE$ and $PE \equiv s$. Hence $PX < s$ and therefore, by what has already been established, X is within Δ_P. This same conclusion would, of course, have been reached had X been in $\measuredangle FOP$. Hence, if X is on the arc P_1PP_2, it lies within the triangle Δ_P. It is true then that every circle is a Jordan curve.

Notation. $\mathfrak{A}_r$ means a circle whose center is A and whose radius is congruent to r.

Theorem 3. Any point P in the plane of the circle $\mathfrak{O}_r$ is within, on, or without $\mathfrak{O}_r$, according as $OP < r$, $OP \equiv r$, or $OP > r$.

Proof. If P is any point in the plane of $\mathfrak{O}_r$ such that $OP > r$, then, with the help of Theorem 5 of § 4, it may be seen that the straight line which lies in the plane of $\mathfrak{O}_r$ and is perpendicular to OP at the point P has no point in common with $\mathfrak{O}_r$. Hence, by the theory of Jordan curves, it follows that P is without $\mathfrak{O}_r$. With the help of O, Definition 7 and Theorem 5 of § 4, it

may be seen that all points P such that $OP < r$ lie in one of the regions into which $\mathfrak{D}_r$ divides its plane. But it has already been established that all points such that $OP > r$ lie in the outside region, and moreover the two regions, into which $\mathfrak{D}_r$ divides all those points of its plane which do not lie on it, are non-vacuous and, finally, for every one of these points, either $OP < r$ or $OP > r$. Hence all points P such that $OP < r$ must lie within $\mathfrak{D}_r$.

Theorem 4. If A and A' are two points and r and r' are two segments such that $r + r' > AA'$ but either $r = r'$ or $r - r' < AA'$, then, if $\mathfrak{A}_r$ and $\mathfrak{A}'_r$ are coplanar, they have two points in common, one on each side of AA'.

Proof. $\mathfrak{A}_r$ has two points, D and E, in common with the straight line AA'. It follows, by hypothesis, Theorems of § 1, Definition 9 and Theorem 2, that one of these points is within and the other is without $\mathfrak{A}'_r$. But the points D and E are joined by two distinct segments of the Jordan curve $\mathfrak{A}_r$, one of these segments lying entirely on one side and the other one entirely on the opposite side of DE. Hence, since $\mathfrak{A}'_r$ also is a Jordan curve, $\mathfrak{A}_r$ and $\mathfrak{A}'_r$ intersect in two points, one on each side of DE.

Theorem 5.* If ABC is any angle and $A'B'$ is any ray, then in any plane containing $A'B'$ there exists, on each side of the straight line $A'B'$, a point C' such that $\measuredangle BAC \equiv \measuredangle B'A'C'$.

Proof. On ray $A'B'$ there exists, by Theorem 1 of § 1, a point B'' such that $AB \equiv A'B''$. By hypothesis, Theorem 6 of § 4 and its corollary, and Theorem 4 of § 5, it follows that, on each side of $A'B'$, there is a point C'' such that $AC \equiv A'C''$ and $BC \equiv B'C''$. But $AB \equiv A'B''$. Hence, by Definition 3, $\measuredangle BAC \equiv \measuredangle B'A'C'$.

If P_0 is assumed in addition to O, C, and K, then one has the following two theorems (6 and 7).

Theorem 6.† If b is any straight line of any plane β then there is some point B, in the plane β but not on b, such that through B there is not more than one parallel to b.

Proof. According to P_0, there exists a straight line b' and a point B' such that through B' there is not more than one parallel to b'. On b' there exist two points C' and D'. If b is any straight line in any plane β then on b there exist, by O and C_{1_a}, two points C and D such that $C'D' \equiv CD$. Furthermore, by Theorem 5 of § 5, there exists in the plane β a point B'' such that $\measuredangle D'C'B' \equiv \measuredangle DCB''$. By Theorem 1 of § 1, there exists on ray CB'' a point B such that $C'B' \equiv CB$. Suppose that through B there are two parallels to b. Then one of these has a point E and the other a point F on that side of BC on which D lies. By Theorem 5 of § 5, there exist in the plane $D'C'B'$,

* See HILBERT's IV, 4.

† See VEBLEN's XII. It may here be remarked that VEBLEN's form of statement lacks the proviso that B should not lie on b.

and on that side of $B'C'$ on which D' lies, two points, E' and F', such that $\measuredangle CBE \equiv \measuredangle C'B'E'$ and $\measuredangle CBF \equiv \measuredangle C'B'F'$. By Theorem 13 of § 1, and A_{-2}, the straight line $B'F'$ must be distinct from the straight line $B'E'$. But neither of these straight lines can have a point in common with $C'D'$. For suppose the ray $B'F'$ and the straight line $C'D'$ had a point G' in common. G' would evidently lie on the ray $C'D'$. By Theorem 1 of § 1, there would exist, on the ray CD, a point G such that $C'G' \equiv CG$. Then one would have $C'B' \equiv CB$, $\measuredangle G'C'B' \equiv \measuredangle GCB$ and $C'G' \equiv CG$. Hence, by Theorem 17 of § 1 and Definition 5, it would be true that $\measuredangle C'B'G' \equiv \measuredangle CBG$. But, by construction, $\measuredangle CBF \equiv \measuredangle C'B'G'$. Hence, by Theorem 13 of § 1 and A_{-2} (which by Theorem 1 of § 2 is a consequence of O, C, and M) the ray BF would be the same as the ray BG. Hence ray BF would have a point in common with ray CG, contrary to hypothesis. In a similar manner, with the help of Theorem 15 of § 1, it could be shown that the other ray of the straight line $B'F'$ could not have a point in common with $C'G'$. Hence the straight lines $B'F'$ and $C'D'$ have no point in common. Similarly, $B'E'$ and $C'D$ have no point in common. But then there exist, through B', two parallels to b', and this is contrary to hypothesis. Thus the assumption that through the point B there are two parallels to b would lead to an absurdity. Hence through B there is not more than one parallel to b.

Theorem 7. If a is any straight line and A any point not on it, then through A there is one and only one straight line parallel to a.

Proof. See Theorem 43 of VEBLEN's "*A System of Axioms for Geometry.*"

In view of Theorems 1, 4 and 6 of the present section, C_2, Theorems 1, 3, 5, 12, 14, 17 of § 1 and Theorem 1 of § 2, it is clear that from O, C, K, P_0 follows a geometry in which HILBERT's axioms of groups I–IV hold true. One may then proceed, as is, for example, indicated in HILBERT's *Festschrift* and HALSTED's *Rational Geometry*, to derive a theory of proportion, etc., and then, with the use of K, one may finally develop an analytic geometry exhibiting a one to one correspondence between the points of our geometry and the number triples of the real continuous number system, this correspondence being such as to preserve all relations of congruence and order. Thus would be established the following theorem:

Theorem 8. From O, C, K and P_0, follows Euclidean geometry.

In case the contradictory of P_0 were assumed in addition to O, C and K, then in place of Theorem 7 one would have Theorem 39 of VEBLEN's *A System of Axioms for Geometry* and again a correspondence could be established between points and number triples, this correspondence also being such as to preserve all relations of order and congruence.* The following theorem would then be established:

* In this case congruence and order for this system of number triple would of course be introduced by definitions different from those used in the case of the correspondence with Euclidean geometry.

Theorem 9. From O, C, K and the denial of P_0 follows Bolyai-Lobachevskian geometry.

It is thus seen that if O, C and K are true of a space, then that space must be either Euclidean or Bolyai-Lobachevskian.

§ 6. *Semi-quadratic geometry.*

In this section will be given several sets of assumptions, any one of which sets is sufficient for the establishment of semi-quadratic or plane semi-quadratic geometry. By semi-quadratic geometry is meant the set of propositions which follow from HILBERT's I–IV. In the case of such a geometry rigid motion is possible, there is a coördinate system and a theory of proportion; and, speaking in terms of this theory of proportion, if $a_1, a_2, a_3, \cdots, a_n$ are any finite number of segments and $F(a_1, a_2, a_3, \cdots, a_n)$ is any rational function of these segments, then there exists a segment equal to $F(a_1, a_2, a_3, \cdots, a_n)$ and also a segment equal to $\sqrt{a^2 + b^2}$.*

Use will be made of the following additional notations for propositions:

R_A (equality of right angles). *If* ABD, $A'B'D'$, $\measuredangle ABC \equiv \measuredangle CBD$ *and* $\measuredangle A'B'C' \equiv \measuredangle C'B'D'$, *then* $\measuredangle ABC \equiv \measuredangle A'B'C'$.

By R_A^{pl} is meant the proposition R_A with the proviso that the angles ABC and $A'B'C'$ are coplanar.† This superscript, "pl," may sometimes be used in notations for other propositions in order that their application may be similarly restricted to the case of coplanar points.

D_{sh} (shortest distance proposition).‡ *If* A, B *and* C *are non-collinear,* ACB', $CB \equiv CB'$, *and* B'' *is a point on ray* AC *such that* $AB \equiv AB''$, *then* $AB''B'$.

P_{er} (perpendicular to a straight line).§ *If* ABC, *then in any plane containing these points there exists a point* D *such that* $\measuredangle ABD \equiv \measuredangle DBC$.

A_*.‖ *If* A, B *and* C *are three non-collinear points and* A', B' *and* D' *are three non-collinear points and* $BA \equiv B'A'$, *then in the half-plane* $D'A'B'$ *there exists one and only one point* C' *such that* $AC \equiv A'C'$ *and* $BC \equiv B'C'$.

A_G¶. *If* $\measuredangle ABC$ *and* $\measuredangle A'B'C'$ *are two angles and* $BA \equiv B'A'$, *then either there is a point* C'' *on the ray* BC *such that* $A'C' \equiv AC''$ *and* $B'C' \equiv BC''$, *or there is a point* C''' *within* $\measuredangle ABC$ *such that* $A'C' \equiv AC'''$ *and* $B'C' \equiv BC'''$, *or there is a point* C^{iv} *within* $\measuredangle A'B'C'$ *such that* $AC \equiv A'C^{iv}$ *and* $BC \equiv B'C^{iv}$; *but for these given angles no two of these statements can both be true.*

P_2. *In every plane* α *there is a straight line* a *such that if* A *is any point of*

* See HILBERT's *Grundlagen der Geometrie.*

† See Theorem 1 of § 4.

‡ See Theorem 6 of § 4.

§ For a stronger proposition see Theorem 10 of § 3.

‖ Cf. Theorem 2 of § 4 and A_{-2} (see § 2). Cf. also HILBERT's IV, 4.

¶ If $\measuredangle ABC$ and $\measuredangle A'B'C'$ are two angles then either $\measuredangle A'B'C' \equiv \measuredangle ABC$ or $\measuredangle A'B'C' > \measuredangle ABC$ or $\measuredangle A'B'C' < \measuredangle ABC$ but no two of these cases can occur simultaneously. Cf. Definition 6, also Theorem 2 of § 3 and corollary to Theorem 2 of § 4.

α not on a, then through A and lying in α there is not more than one straight line which has no point in common with a.

Theorem 1. From O^{pl}, C, M, and P_2 follows plane semi-quadratic geometry.

Proof. See Theorems 1, 3, 5, 12, 14, 17 of § 1 and Theorems 1 and 2 of § 2 and Theorem 2 of § 4. The proposition that through each point there is one and only one parallel to any given straight line may be proved with the help of P_2 by a method which is suggested in proof of Theorem 6 of § 5.

Theorem 2. From O^{pl}, C, A_{-2}, A_1, P_2, follows plane semi-quadratic geometry.

Proof. In the proof above indicated for Theorem 1, M is used only to demonstrate A_{-2} and A_1.

Lemma 1. From O^{pl}, C and D_{sh} follows A_{-2}.

Proof. Suppose O, C and D_{sh} are true and A_{-2} is not. Then there exist three non-collinear points A, B, C, and a point C', all in the half-plane $C-AB$, such that $AC \equiv AC'$ and $BC \equiv BC'$. According to O, there exist points A', B' such that ACA', BCB'. There are five cases.

I. If C' is in $\measuredangle ACB'$, then segments AC and BC' have a point O in common. According to D_{sh}, $AO + OC' > AC'$ and $OC + BO > BC$. Thus $(AO + OC) + (OC' + BO) > BC + AC'$. Hence $AC + BC' > BC + AC'$; this is impossible in view of the hypothesis that $AC \equiv AC'$ and $BC \equiv BC'$.

II. If C' is in $\measuredangle BCA'$, argue in a similar manner.

III. If C' is in $\measuredangle ACB$, then there exists a points D such that $AC'D$, BDC. Now $AC + CD > AD$. Hence $AC + CD + DB > AD + DB$, $AC + CB > AC' + C'D + DB$. But $C'D + DB > C'B$. Therefore $AC + CB > AC' + C'B$, and this is impossible in view of hypotheses.

IV. If C' is in $\measuredangle A'CB'$, proceed as in case III.

V. If C' is on the straight line AC or the straight line BC, proceed with C or C' as in case III with D.

Thus in any case it would be impossible that O, C and D_{sh} should be true and A_{-2} false simultaneously.

Lemma 2. From O, C and P_{er} it follows that through any point there is at least one perpendicular to any given straight line.

Proof. See the proof of Theorem 10 of § 3.

Lemma 3. A_{-2} is a consequence of O, C and P_{er}.

Proof. Suppose C, A and B are three non-collinear points and C' is a point in the half-plane $C - AB$ such that $CA \equiv C'A$ and $CB \equiv C'B$. By Lemma 2, there is, on the straight line AB, a point D such that CD is perpendicular to AB. In case D coincides with A, then $\measuredangle BAC$ is a right angle and therefore, by hypothesis and corollary to Definition 7 of § 3, $\measuredangle BAC$ is a right angle and hence, by P_{er} and C_{1_b}, C' is C. If D does not coincide with A, then, by Theorem 17 of § 1, $\measuredangle ADC \equiv \measuredangle ADC'$ and $DC \equiv DC'$. But

$\measuredangle ADC$ is a right angle. Hence, by the corollary to Definition 7 of § 3, $\measuredangle ADC'$ is a right angle, and therefore, by P_{cr}, ray DC' coincides with ray DC. But $DC \equiv DC'$. Hence, by C_{1_b}, C' is C.

Theorem 3. From O^{pl}, C, A_1, D_{sh} and P_2 follows plane semi-quadratic geometry.

Proof. See Lemma 1 and Theorem 2.

Theorem 4. From O, C, A_s and P_2 follows semi-quadratic geometry.

Proof. Compare Theorems 1, 3, 5, 12, 14, 17 of § 1 and Definition 3 and A_s with HILBERT's group IV of axioms. Use argument, concerning parallel proposition, suggested in proof of Theorem 1 of § 6.

Theorem 5. From O, C, A_{-2}, R_A and P_2 follows semi-quadratic geometry.

Proof. It may be seen, from the proof of Theorem 2 of § 4, that A_s is a consequence of O, C, A_{-2} and R_A. But, by Theorem 4 of § 6, from O, C, A_s and P_2 follows semi-quadratic geometry.

Theorem 6. From O, C, P_{cr}, R_A, P_2 follows semi-quadratic geometry.

Proof. See Lemma 3 and Theorem 5.

Theorem 7. From O, C, D_{sh}, R_A and P_2 follows semi-quadratic geometry.

Proof. See Lemma 1 and Theorem 5.

Theorem 8. From O, C, A_G and P_2 follows semi-quadratic geometry.

Proof. According to Theorem 4, this theorem will be established if it is shown that A_s is a consequence of O, C and A_G.

Suppose A, B and C are three non-collinear points and D', B' and E' are three non-collinear points. On the ray $B'D'$ there is a point A' such that $BA \equiv B'A'$. To prove that in the half-plane $E' - A'B'$ there is one and only one point C' such that * $BC \equiv BC'$ and $AC \equiv A'C'$, argue as follows. According to A_G either (I), there is a point C', on $B'E'$ or in $\measuredangle A'B'E'$, such that $BC \equiv B'C'$ and $AC \equiv A'C'$, or (II), there is a point C'' within $\measuredangle ABC$, such that $\measuredangle ABC'' \equiv \measuredangle A'B'E'$. In this last case there is, on the ray BC'', a point F such that AFC. By Theorem 11 of § 1 there exists on ray $B'E'$ a point F' such that $AF \equiv A'F'$ and $BF \equiv B'F'$. By C_{1_a} there is a point C' such that $A'F'C'$ and $FC \equiv F'C'$. By C_3 and Theorem 11 of § 1, $AC \equiv A'C'$ and $BC \equiv B'C'$.

Suppose there were, in the half-plane $E' - A'B'$, two points C' and C'' such that $AC \equiv A'C'$ and $BC \equiv B'C''$. Then, by C_{1_a} and O, either C'' is within $\measuredangle A'B'C'$ or C' is within $\measuredangle A'B'C''$. Suppose, for instance, C'' is within $\measuredangle A'B'C'$. Then, by hypothesis, Definition 3 and Theorems 11 and 18 of § 1, there exists within $\measuredangle ABC$ a point C''' such that $B'C'' \equiv BC'''$ and $A'C'' \equiv AC'''$. But, by hypothesis, C_2 and Theorem 4 of § 1, $A'C' \equiv A'C'$ and $B'C' \equiv B'C''$. Hence, by C_2, $A'C' \equiv AC'''$ and $B'C' \equiv BC'''$. But C'' is within $\measuredangle A'B'C'$ and $AC \equiv A'C''$ and $BC \equiv B'C''$. Thus the

* Cf. A_s, page 503.

supposition that there are two points C' and C'', in the half plane $E'A'B'$ such that $AC \equiv A'C'$, $BC \equiv B'C'$, $AC \equiv A'C''$ and $BC \equiv B'C''$, would lead to a contradiction with A_G.

Theorem 7 is therefore established.

§ 7. *Geometry of the rule and compasses.*

In this paragraph will be considered several sets of assumptions, from any one of which sets follows a geometry in which all ordinary rule and compasses constructions are possible.

Two propositions concerning intersections of straight lines with circles, or of circles with each other, will be considered and will be referred to by means of the following notations:

I_1.* *If A, B and C are non-collinear points and P is between A and B, then there is, on the straight line DC, a point B' such that $AB \equiv AB'$.*

I_2.† *If APD, PBP', $AD'P'$, $PB \equiv BP'$, $AD \equiv AD'$, and all of these points lie in a plane, and this plane is decomposed by the straight line PB into two regions,‡ then, in each of these regions, there is a point C such that $AC \equiv AD$ and $BC \equiv BP$.*

Lemma 1. M is a consequence of O, C, and I_2.

Proof. Suppose A and B are two different points. From O and C it easily follows that there is a point P, between A and B, such that $AP \succ PB$. Then there is, by Theorems 3 and 6 of § 1, a point P' such that $AP'B$, $AP' \equiv PB$, and $P'B \equiv AP$. Evidently $AP'P$. Hence, by I_2, there exist two points D and D', on opposite sides of AB, such that

$$BD \equiv AD \equiv BD' \equiv AD' \equiv AP.$$

Segment DD' meets the straight line AB in a point M. According to Definitions 3, 5 and Theorems 3 and 17 of § 1, $MA \equiv MB$. Thus every segment has a middle point.

Lemma 2. I_2 is a consequence of O, C, M, P_2 and I_1.

Proof. According to Theorem 1 of § 6, plane semi-quadratic geometry is a consequence of O^{pl}, C, M, P_2. So a theory of proportion and an analytic geometry may be introduced. Now suppose one extremity of a diameter of $\mathfrak{C}'_{r'}$ is within $\mathfrak{C}_r$ and its other extremity is without $\mathfrak{C}_r$. Take CC' as x-axis and a perpendicular to CC' at the point C as y-axis. Equations of $\mathfrak{C}_r$ and $\mathfrak{C}'_{r'}$, referred to these axes, are $x^2 + y^2 = r^2$ and $(x - a)^2 + y^2 = r'^2$ respectively, where a is the abscissa of C'. It is evident, in view of I_1, that for every value,

* If a straight line lies in the plane of a circle and has a point within that circle then it intersects the circle.

† If a semicircle has one extremity, P, within and the other, P', without a circle with which it is coplanar, then it has a point in common with that circle.

‡ Cf. VEBLEN, loc. cit., pages 363 and 364.

x', of x between $-r$ and $+r$, there is a value, y', of y such that (x', y') satisfies $x^2+y^2=r^2$. With help of the fact that $r+r'>a$, $a+r>r'$ and $a+r'>r$, it may be seen that $\frac{1}{2}a^{-1}(r^2+a^2-r'^2)$ is between $-r$ and $+r$. Take for x' this value. Then

$$y'=\pm\sqrt{r'^2-x'^2}$$

and, no matter which sign we take, these values of x' and y' evidently satisfy $(x-a)^2+y^2=r'^2$ as well as $x^2+y^2=r^2$. Thus $\mathfrak{C}_r$ and $\mathfrak{C}'_{r'}$ intersect in two points, one on each side of CC'.

Lemma 3. I_1 is a consequence of O, C and I_2.

Proof. From O, C and I_2, M follows, according to Lemma 1. Now suppose the straight line p lies in the plane of the circle $\mathfrak{C}_r$ and passes through a point P which is within that circle. If P is the same as C, then, by Theorem 1 of § 1, p evidently has a point in common with $\mathfrak{C}_r$. If P is different from C, it follows from O, C and M (see Theorem 10 of § 3), that there exists on p a point D such that CD is perpendicular to p. By C_{1_a} there is a point C' such that CDC' and $CD\equiv DC'$. Then, evidently, according to hypothesis, Theorem 5 of § 4, Theorems 1 and 2 of § 2, and I_2, $\mathfrak{C}_r$ and $\mathfrak{C}'_r$ have a point F in common. Hence, according to Definitions 3 and 7, FD is perpendicular to CD. Hence, by Theorem 10 of § 3, F lies on the straight line p. So p and $\mathfrak{C}_r$ have a point F in common.

Theorem 1. From O, C, I_2, and P_2 follows a "Geometry of the Rule and Compasses."

Proof. With the help of Theorem 1 of § 6 it may easily be seen that a geometry of the rule and compasses for each plane follows from O, C, M, P_2 and I_2. Our present theorem will then be established if it is proved, (I), that M follows from O, C, P_2 and I_2, and, (II), that a geometry of the rule and compasses follows from I_3, P_2, O, and geometry of the rule and compasses for each plane. Now M follows from O, C, P_2 and I_2, according to Lemma 1; and (II) is evident in view of the fact that Theorem 5 of § 5 is proved as a logical consequence of I_2 and plane semi-quadratic geometry for each plane.

§ 8. *Independence of each assumption in the set composed of K, P_0, C and O.*

The following "independence examples" are constructed to show that each assumption of the set composed of K, P_0, C, and O is independent of the remaining ones.

Example for order Axiom I. Use VEBLEN'S K_{I} (loc. cit., page 353).

Example for order Axiom III. Consider four points A, B, C, D in the orders ABC, ACB, ADB, BAD, BCA, BDC, CAD, CBD, CDA, DAC, DBA, DCB. Consider that every segment $\equiv$ every segment.

Example for order Axiom IV. Use VEBLEN'S K_{IV}. Consider that every segment $\equiv$ every segment.

Example for order Axiom VI. Points are all integers (including 0). $0K4$ and $4K0$ for every point K except -1, 0, 4 and 5. -1, 0 and 4 are in the orders $-1\,0\,4$ and $4\,0\,-1$. 0, 4 and 5 are in the orders $0\,4\,5$ and $5\,4\,0$. If, in the ordinary sense, $0<A<4$, $C>4$ and $0<B<5$, then $\{^{ABC}_{CBA}\}$ if and only if $B-A=C-B-1$. If, in the ordinary sense, $0<A<4$, $C<0$ and $-1<B<4$, then $\{^{ABC}_{CBA}\}$ if and only if $A-B=B-C-1$. If D and E are two integers and there is no integer C such that DEC according to these definitions given above, then $\{^{DEC}_{CED}\}$ if and only if $E-C=D-E$. Every segment $\equiv$ every segment.

According to this plan one has the following orders:

8 4 1	1 3 4	-4 0 3	3 0 -4
7 4 2	2 4 7	-3 0 2	2 0 -3
6 4 3	3 4 6	-2 0 1	1 0 -2
6 3 1	1 3 6	-2 1 3	3 1 -2
5 4 0	0 4 5	-1 0 4	4 0 -1
5 3 2	2 3 5	-1 1 2	2 1 -1

$0K4$ and $4K0$ for every integer $K\neq -1, 0, 4$ or 5; and in case A and B are two integers which are not respectively the first and second element of any one of these triads, then ABC if and only if $A-B=B-C$.

Example for order Axiom VII. VEBLEN'S K_{VII}. Every segment $\equiv$ every segment.

Example for order Axiom VIII. VEBLEN'S K_{VIII}. Every segment $\equiv$ every segment.

Example for order Axiom IX. VEBLEN'S K_{IX}. Congruence ordinary.

Example for order Axiom X. VEBLEN'S K_{10}. Congruence ordinary.

Example for order Axiom XI or K. Some non-Archimedean geometry, for example that of HILBERT (cf. TOWNSEND'S translation of HILBERT'S *Grundlagen der Geometrie*, page 34).

Example for P_0. Consider any of the ordinary proofs of the compatibility of Bolyai-Lobachevskian geometry.

Example for C_{1_a}. Define points as the points of ordinary Euclidean space that lie on one side only of a given plane, consider these points to be ordered just as in the ordinary sense and two segments to be congruent to each other if and only if they are congruent in the ordinary sense. Then K and P_0 are satisfied and so are all the assumptions of C except C_{1_a}.

Example for C_{1_b}. Consider points to be all the points of ordinary Euclidean space ordered in the ordinary manner, but consider every segment as being congruent to itself and every other segment. Evidently C_{1_b} is the only assumption of K, P_0 and C that is not satisfied in this example.

Example for C_2. Consider points to be all the points of ordinary Euclidean

space ordered in the usual manner, but consider that every segment is congruent to every segment which in the ordinary sense is just twice as long.

Example for C_3. Consider points to be the points of ordinary Euclidean space ordered in the usual manner, but consider that there is one segment AB such that $CD \equiv C'D'$ if and only if $C'D'$ is congruent to AB in the ordinary sense.

Example for C_4. Consider points to be the points of ordinary Euclidean space ordered in the usual manner. Consider a certain fixed plane p and regard two segments neither of which is perpendicular to that plane as being congruent to each other if and only if their orthogonal projections upon that plane are congruent to each other in the ordinary sense; in case only one of them is perpendicular to p, regard them as congruent if and only if this one is congruent in the ordinary sense to the projection of the other one upon p; finally, if they are both perpendicular to p, regard them as congruent if and only if they are congruent in the ordinary sense.

As was suggested by Professor E. H. MOORE, the independence example here given for C_2 does not prove C_2 independent of K, C and the negative of P_0. But all of the other independence examples here given (except, of course, that for P_0) do apply if the negative of P_0 is substituted for P_0 and Bolyai-Lobachevskian space is used instead of Euclidean. Of course the negative of P_0 is shown to be independent of O, C and K if point, order and congruence are taken as those of ordinary Euclidean space.

§ 9. *Independence of each postulate in the set composed of* I_3, P_2, C *and* O.

Except for C_{1_a} and I_2 use the same examples as in § 8.

To prove C_{1_a} independent, observe that all the assumptions P_0, C, I_2, except C_{1_a}, are satisfied, either "vacuously" or otherwise, if "points" are all the points of ordinary Euclidean space ordered as usual, no segment, however, being congruent to any segment. C_2, C_3, C_4 and I_2 are, in this case, "satisfied vacuously."

To prove I_2 independent, consider the space obtained by omitting all the points of Euclidean space except those whose coördinates are rationally expressible in terms of expressions of the form $\sqrt{r}$ where r is an integer.

§ 10. *Relation of parallel assumptions to introduction of congruence by definition.*

With use of O, K (or order "Axiom XI"), and VEBLEN's Axiom XII* (concerning parallels), congruence may† be introduced by definition, so that, if O, K and XII are true of a space, then there must exist between the segments of that space a relation satisfying, for example, all of my assumptions C. But

*See footnote † on page 501.

† VEBLEN, loc. cit., page 383.

this is not true if XII is replaced by the weaker postulate P_0, as may be seen with help of the independence example obtained by considering a certain fixed plane in ordinary Euclidean space and regarding as "points" only those points of this space which are on a certain side of this fixed plane, these points being ordered as usual. Here O, K (also XI) and P_0 are satisfied. But there cannot exist among the segments of this space a relation satisfying assumptions C. For in that case this space would be, according to § 5, ordinary Euclidean space and thus, in particular, through each point outside of any given line there would be only one parallel to that line, as is manifestly impossible.

§ 11. *Relation of continuity assumptions to introduction of congruence by definition.*

Theorem 1. It is not possible to prove that if the assumptions* O, P_4, C_{1_b}, C_2, C_3, C_4 (call this the set S), hold true of a space, then there exists a definition for the symbol "$\equiv$" such that the assumptions C and M will hold true with reference to the points and order in terms of which the assumptions of S are stated.

Proof. Select any system (OX, OY, OZ) of three rectangular axes in ordinary Euclidean space and consider the space composed of all points whose coördinates with reference to this system are all rational, congruence and order relations being as usual. Call this space U and let the terms "original order" and "original congruence" be understood here as meaning the order and congruence here indicated for this space. Clearly O, P_4 and all of the assumptions of C except C_{1_a} hold true of U with reference to this order and congruence. Suppose a new meaning could be given to the congruence symbol "$\equiv$" such that the assumptions C and M would hold true for the space U with reference to the original order. Then, according to § 6, all the theorems of plane semi-quadratic geometry, in particular a theory of proportion, would hold true, with reference to this new congruence and the original order, for any planes in the space U. There would then exist on OX a point P such that, with reference to this theory of proportion, $OP \times OP = 2OI$ where I is some point of OX (e. g., such that $OI =$ original unit). OP would then clearly not be, with reference to this new congruence, rational in terms of OI. But with use of P_4 it may be shown that any two segments which lie on OX, being rational in terms of each other with reference to the original congruence, would necessarily also be so with reference to this new congruence. To prove this let us first suppose that AB and CD are two segments of OX and $AB \equiv CD$ according to original congruence. By P_4 there exists a parallelogram $ABEF$ in which AB and EF

* By P_4 is meant the strong parallel assumption: If a is any straight line and A is any point not on a then, in the plane aA, there is one and only one straight line which passes through A and has no point in common with a.

are opposite sides. Manifestly $EFCD$ will be a parallelogram in which EF and CD are opposite sides. But it is clear that, in terms of our new as well as in terms of our old congruence, the opposite side of a parallelogram must be congruent to each other. Therefore, in terms of this new congruence, $AB \equiv EF$ and $EF \equiv CD$ and therefore $AB \equiv CD$. It is true then that if AB and CD are segments of OX and $AB \equiv CD$ according to old congruence, then also $AB \equiv CD$ according to our new congruence. Now suppose that the segments AB and CD, of the line OX, are given as rational in terms of one another (instead of simply congruent to one another) in terms of original congruence. Then there exists a segment KL, on OX, such that AB can be divided into m segments and CD can be divided into n segments, all congruent in the original sense, to KL, m and n being positive integers. But these $m + n$ segments of OX, being congruent to each other according to the original congruence, must also be congruent according to our new congruence, and thus AB and CD are rational in terms of one another according to new congruence. But it has been shown that, were such a new congruence possible, then there would exist on OX two segments, OP and OI, which would not be rational in terms of one another according to this congruence. A contradiction would thus be obtained and such a congruence would therefore be impossible.

§ 12. *A question concerning the separation of the assumptions for a geometry into two sets.*

Theorem 1. It would be impossible to formulate a set (I) of assumptions expressed in terms of point and order alone and a set (II) of assumptions expressed in terms of point and congruence alone, such that any geometry satisfying the assumptions I and II must necessarily be ordinary Euclidean geometry with respect to the undefined symbols in terms of which I and II are stated.

Proof. Consider a system of three rectangular axes OX, OY, OZ in an ordinary Euclidean space (E). Consider a paraboloid of revolution (G) whose axis is OZ and whose vertex is O. Let order be as usual but consider that segment $AB \equiv$ segment $A'B'$ if and only if

$$(x_A - x_B)^2 + (y_A - y_B)^2 + (q_A - q_B)^2 = (x_{A'} - x_{B'})^2 + (y_{A'} - y_{B'})^2 + (q_{A'} - q_{B'})^2,$$

the new coördinate, q_P, being defined as 0 if P is on G, and otherwise as $\pm$ the length of SP where S is the point in which G is cut by a parallel to OZ through the point P, and $+$ or $-$ is used according as P is on the same side of the surface G as is Z, or on the other side. Let C' designate this particular congruence. It may be seen that any statement concerning point and order alone that is true for Euclidean space must hold true for the space E with reference to the congruence C'. For if a straight line were defined as the locus of a point P

satisfying two linear equations of the first degree in x_P, y_P, q_P, and, in accordance with this, a new order O' were defined in the usual analytical manner, then one would have a Euclidean geometry concerning the points of the space E, the congruence C' and the order O'. But also, of course, any statement concerning points and order alone which is true of ordinary Euclidean space is true of our original order for the particular space E. But manifestly space E does not satisfy the theorems of ordinary Euclidean geometry with reference to ordinary order and the congruence C'.

§ 13. *Certain queries.*

I do not know as yet whether M* is a consequence of O† and C†. There are several other questions which could not be settled negatively without deciding this question. For example: Is D_{sh}‡ a consequence of O, C, A_1§ and P_2?‡ Is A_1 a consequence of O, C, D_{sh} and P_2? Is R_A‡ a consequence of O, C, A_{-2}§ and P_2? Is A_{-2} a consequence of O, C, R_A and P_2? Is A_1 a consequence of O, C, A_{-2} and P_2? Is A_{-2} a consequence of O, C, A_1 and P_2?

Another question is whether Theorem 1 of § 11 would be true if C_{1_a} were substituted for either C_{1_b}, C_2, C_3 or C_4.

In § 6 it was shown that from O, C, M and P_2 follows a geometry for every plane of which all the theorems of plane semi-quadratic geometry hold true. Is then the semi-quadratic geometry of a three-space a consequence of O, C, M and P_2? This could be answered in the affirmative if it could be shown that in every space for which O, C and M are true all right angles are congruent to each other. It can be proved‖ that in such a space all *coplanar* right angles are congruent to each other.

* See last sentence of § 1.

† See the opening pages of the paper.

‡ See § 6.

§ See § 2.

‖ See Theorem 1 of § 4.